RESEARCH STATEMENT

TERRENCE GEORGE

My research lies at the intersection of combinatorics, probability, and algebraic geometry. I have explored combinatorial models in statistical mechanics—specifically the dimer model, electrical networks, and the Ising model—and their connections to discrete geometry, integrable systems, algebraic geometry, etc.

The **dimer model** was originally studied in statistical physics as a model for the adsorption of diatomic molecules on crystal surfaces. It can also be interpreted as a model for a melting crystal. Mathematically, a **dimer cover** (or **perfect matching**) of a graph is a subset of the edges that use every vertex exactly once (Figure 1). The enumeration of perfect matchings is a classical problem in graph theory and is known to be computationally hard. A celebrated result of Kasteleyn [Kas63] and of Temperley and Fischer [TF61] says that, for planar bipartite graphs, the number of dimer covers can be computed in polynomial time as the determinant of a version of the adjacency matrix of the graph, called the **Kasteleyn matrix**. Underpinning the success of the dimer model is this determinantal structure, which effectively reduces probabilistic computations for the dimer model to linear algebra.

Beyond statistical mechanics, recent research has uncovered surprising and profound connections between the determinantal structure of the dimer model and various fields, including total positivity in combinatorics, integrable aspects of discrete differential geometry, discrete complex analysis and graph embeddings, and Harnack curves and Beauville integrable systems in algebraic geometry. Through my work, I aim to contribute to a deeper understanding of these relationships and the broader implications of the dimer model across these various fields, with a particular emphasis on integrating geometric and algebraic methods.

1. Cluster integrable systems

Many of the connections of the dimer model to other areas of mathematics involve moduli spaces of weighted bipartite graphs on surfaces, which possess a rich underlying structure. These spaces are instances of Fock and Goncharov's **cluster varieties** [FG09], the geometric counterparts of Fomin and Zelevinsky's cluster algebras [FZ02]. A cluster variety is

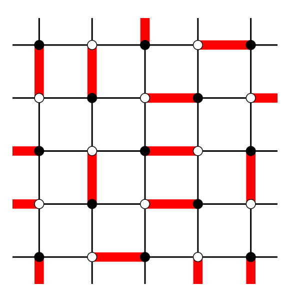


Figure 1. A dimer cover of a portion of the square lattice graph that models the surface of a crystal with two types of molecules corresponding to black and white vertices.

constructed by gluing algebraic tori using maps called **cluster mutations** and is equipped with a canonical Poisson structure [GSV10]. The dimer model has a local transformation called the **spider move**, known from the beginnings of cluster algebras as an example of a cluster mutation, with the associated cluster variety parameterizing the space of dimer models.

Building on these observations, Goncharov and Kenyon [GK13] constructed a cluster variety of dimer models on a torus for each convex integer polygon in the plane and showed that it is a Hamiltonian (Liouville–Arnold) integrable system which they called a **cluster integrable system**. It was later shown in [FM16] that many well-known integrable systems, such as the Toda lattice and the pentagram map, are special cases of cluster integrable systems. The **spectral transform** of Kenyon and Okounkov [KO06] provides action-angle coordinates for the integrable system, a set of coordinates describing a **spectral curve** and a point in its Jacobian, in which the system's time evolution is linear. Fock [Foc15], using the theory of finite-gap solutions of integrable systems, explicitly constructed the inverse of the spectral transform and showed that the spider move is equivalent to Fay's celebrated trisecant identity for Jacobian theta functions. With Alexander Goncharov and Richard Kenyon [GGK23], I gave an alternative construction of the inverse spectral transform that only involves rational functions (as opposed to the transcendental theta functions) and is hence amenable to symbolic computation.

A common theme is that the results for the dimer model have parallels in other statistical mechanical models, such as electrical networks (equivalently, spanning trees) and Ising models. In [Geo24a, Geo24b], I studied the generalization of cluster integrable systems to electrical networks and Ising models. In both cases, the spectral curve is a Harnack curve with an involution. Whenever a curve carries an involution, it determines a linear subvariety of the Jacobian called a **Prym variety**.

Theorem 1.1 ([Geo24a])**.** *Spaces parameterizing toric electrical networks and Ising models are isomorphic to spaces of symmetric spectral curves and their Prym varieties. The electrical star-triangle transformation is equivalent to Fay's quadrisecant identity for Prym theta functions.*

These papers reveal that these models possess many, if not all, of the integrable structures present in the dimer model, and I believe there is still much to be explored in this area. For example, two significant open questions are:

Question 1.2. Do these spaces carry natural Poisson structures and are they integrable in the sense of Liouville–Arnold?

Question 1.3. What is the theta-function identity corresponding to the Ising star-triangle transformation, also known as the discrete CKP equation?

Associated with a convex integer polygon in the plane is another integrable system from algebraic geometry: the **Beauville integrable system** [Bea91, Bea90] on the corresponding projective toric surface. Goncharov and Kenyon [GK13] conjectured that the spectral transform is an isomorphism between integrable systems. After the initial strategy of Goncharov and Kenyon to prove this conjecture through combinatorial methods was unsuccessful, Giovanni Inchiostro and I proved it in [GI22] using homological algebra machinery for the dimer model on the hexagonal lattice. Establishing the equivalence of cluster integrable systems with Beauville integrable systems, in general, would be very interesting. I also believe that these powerful, though abstract, algebraic techniques could have broader applications beyond what we have explored so far.

2. Cluster integrability in discrete differential geometry

Aside from being integrable in the Liouville–Arnold sense, cluster integrable systems have discrete dynamics that are obtained by iterating spider moves called **cluster modular transformations** in [FG09]. A celebrated example is the pentagram map [OST10], a cluster modular transformation on the square lattice. Recently, several other integrable systems have been discovered in discrete differential geometry (see, e.g., [ILP16, AFIT22, Izo23]) that belong to the class of cluster integrable systems. Together with Niklas Affolter and Sanjay Ramassamy, I developed a general framework in [AGR21] to prove the equivalence of these integrable systems with cluster integrable systems. Unlike previous approaches that relied on algebraic methods to compare coordinates, our work introduces a new geometric perspective. This approach, together with [AGR] and [AGPR24], provides a unified understanding of the integrable aspects of discrete differential geometry, dimer models, and cluster algebras. A prototypical application of our theory is:

Theorem 2.1 ([AGR21])**.** *The cross-ratio dynamics integrable system of* [AFIT22] *is a cluster integrable system.*

In addition to geometry, cluster modular transformations also emerge from the combinatorial and probabilistic study of the dimer model. These transformations have been used to compute partition functions, perform exact sampling via shuffling algorithms, and prove limit shape results (see, e.g., [EKLP92, JPS98, PS05, KP16]). Fock and Marshakov [FM16] investigated the group of cluster modular transformations and conjectured its isomorphism type, a conjecture that Giovanni Inchiostro and I confirmed in [GI24]. However, in several of the recently discovered discrete integrable systems, the dynamics is governed not only by sequences of spider moves but also by an additional non-local transformation known as the geometric R-matrix, first introduced by Inoue, Lam, and Pylyavskyy [ILP16]. With Sanjay Ramassamy, I computed the group of discrete dynamics that arises in this context in [GR23], extending the results from [GI24].

3. Total positivity

The **Grassmannian**, the space parameterizing all subspaces of a fixed dimension of a vector space, is a classical object in algebraic geometry. An explicit way to present an element of the Grassmannian is to specify a rectangular matrix whose rows span the subspace. The maximal minors of the matrix provide a set of homogeneous coordinates called **Plücker coordinates**. A totally non-negative matrix is a matrix whose minors are all non-negative. Generalizing this notion, Postnikov [Pos06] defined the **totally non-negative Grassmannian** to be the set of elements of the Grassmannian with all Plücker coordinates non-negative. Roughly, the Kasteleyn matrix of a dimer model on a disk gives an element of the totally non-negative Grassmannian. Postnikov showed that this construction, called **boundary measurement**, is a bijection between the space of dimer models and the totally non-negative Grassmannian. Moreover, degenerations of dimer models induce a beautiful stratification of the totally non-negative Grassmannian into **positroid** cells. Lam [Lam18] showed that boundary measurements of electrical networks form a stratified subspace of the

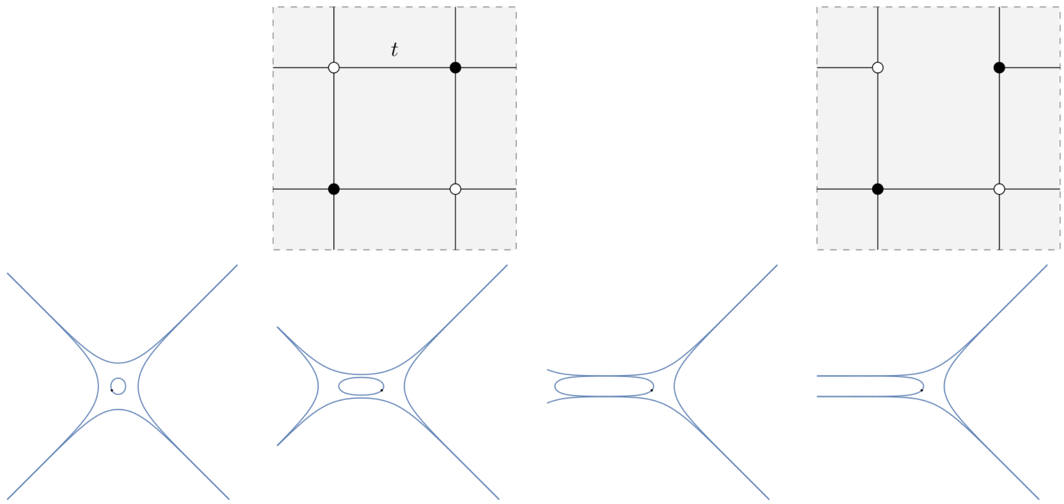


FIGURE 2. Degeneration of a weighted bipartite graph with weights as $t \to 0$, and the corresponding degeneration of the (amoeba of the) Harnack curve and line bundle (drawn as a divisor).

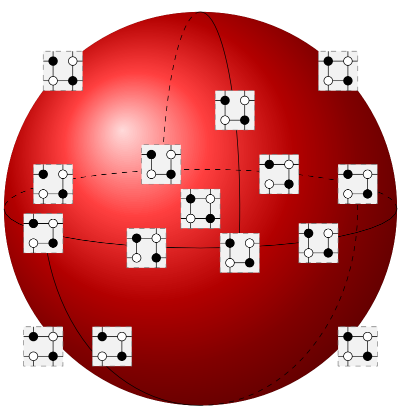
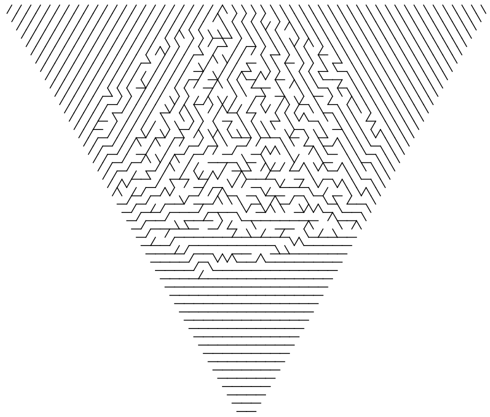


FIGURE 3. An example of a stratified "totally non-negative" space parameterizing degenerations of toric dimer models/Harnack curves and Jacobians. Each cell is labeled by an equivalence class of bipartite graphs, a representative of which is shown.

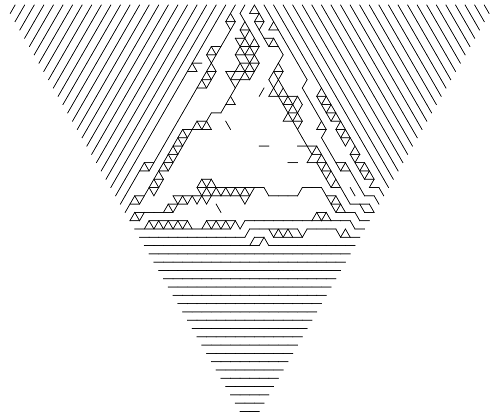
totally non-negative Grassmannian. With Sunita Chepuri and David Speyer, I showed that:

Theorem 3.1. [CGS21] *Lam's subspace is a totally non-negative Lagrangian Grassmannian.*

Together with the work of Galashin and Pylyavskyy on the Ising model [GP20], this completes a compelling classification of totally non-negative Grassmannians associated with statistical-mechanical models in the disk. In [Geo24c], I studied the problem of explicitly reconstructing an electrical network from its boundary measurements. This is a classical problem which has been studied by many authors over the years. I found a new, and in a sense, natural solution using ideas from cluster algebras.



(a) Grove



(b) FK percolation

FIGURE 4. Simulations of random groves and percolation configurations on the triangular lattice showing the emergence of limit shapes. In each picture, there is a "liquid" region in the middle outside which the random object appears frozen or "solid."

4. Total positivity on a torus

Kenyon, Okounkov, and Sheffield [KOS06] proved a surprising and deep connection between toric dimer models and real algebraic geometry. They showed that the spectral curve is a real algebraic curve with a special topology called a **Harnack curve** and used this fact to obtain a universal classification of the phases of the model. Later, Kenyon and Okounkov [KO06] constructed the spectral transform, an isomorphism between the space of toric dimer models and the moduli space of Harnack curves and (a component of) their real Jacobians. The spectral transform is, in a precise sense, the toric analog of Postnikov's boundary measurement. Together with Pavel Galashin, one of my long-term research goals is to understand degenerations of toric dimer models (Figure 2) and the induced positroid-like stratification on the moduli space of Harnack curves and their Jacobians. As a first step, in [GG24], we study the combinatorics of the stratification and determine the analog of positroids:

Theorem 4.1. [GG24] *Toric bipartite graphs modulo moves are classified by integer polygons in the plane with some additional data.*

This is already considerably more complicated and subtle than in Postnikov's setting (Figure 3). The next step would be to study the geometry of the strata, which we presently only understand in small examples.

5. Limit shapes

A fascinating aspect of dimer models is the emergence of **limit shapes**, a manifestation of the law of large numbers, where a random dimer cover of a large graph concentrates around a deterministic shape. The first result of this kind was the celebrated Arctic Circle Theorem for domino tilings [JPS98]. These limit shapes can be interpreted as models for melting crystals in equilibrium and have distinct macroscopic regions corresponding to solid and liquid phases. In electrical networks, the equivalent of dimer covers are spanning-tree-like objects called **groves**. In [Geo21], I used the integrable structure of the model to extend the

Arctic Circle Theorem for groves due to Petersen and Speyer [PS05] (Figure 4(a)). While this technique only identifies the boundary between different phases and not the full limit shape, it shows that groves display similarly rich limit shape behavior as dimers and deserve further study. With Russkikh, Kenyon, and Vu, I am investigating limit shapes in Fortuin–Kasteleyn (FK) percolation models, a one-parameter family of statistical-mechanical models generalizing groves (Figure 4(b)).

6. Other short and medium-term projects

I am excited about working with students on research projects. Below, I outline some potential projects aligning with my research interests and knowledge that would be well-suited for PhD students or highly motivated Masters students.

6.1. **The twist map for electrical networks/Ising models.** Cluster varieties are part of an ensemble $\mathcal{A} \to \mathcal{X}$, comprising two varieties, $\mathcal{A}$ and $\mathcal{X}$, with a canonical map between them. In the case of the dimer model on a disk, both $\mathcal{A}$ and $\mathcal{X}$ are subvarieties of the Grassmannian, called open positroid varieties [KLS13]. The canonical map is an automorphism between these varieties, called the *twist* [MS16, MS17]. Although electrical networks and Ising models are part of similar cluster-like ensembles [GK13, KP16], where the $\mathcal{X}$ varieties are relatively well understood [BGKT23, GP20, Lam18, CGS21], the $\mathcal{A}$ varieties and the twist require further investigation. I worked out the simplest case in [Geo24c].

6.2. **Positive geometry.** An exciting development at the intersection of combinatorics and algebraic geometry, inspired by studying scattering amplitudes in physics, is the emerging field of Positive Geometry [AHBL17]. Lam [Lam16] showed that positroid varieties are instances of positive geometries. It would be interesting to prove that electrical networks and Ising models are also examples of positive geometries. Doing this requires new approaches to study the geometric properties of these spaces, as the representation-theoretic methods used by [KLS13] for positroid varieties appear to be inadequate.

6.3. **Electrical matrix Schubert varieties.** Matrix Schubert varieties, introduced by Fulton [Ful92], are determinantal varieties whose Gröbner degenerations were shown by Knutson and Miller [KM05] to provide a geometric explanation for the positive expansion of Schubert polynomials using pipe dreams. Snider [Sni10] extended this to open patches in positroid varieties and affine pipe dreams. Together with David Speyer, we computed several examples of analogous degenerations of response matrices in electrical networks which are characterized by objects resembling affine pipe dreams, and would be very interesting to investigate further.

6.4. **Limit shapes for cube groves using the spectral transform.** Recently, Boutillier and de Tilière [BdT24] applied Fock's inverse spectral transform [Foc15] for the dimer model to investigate domino tilings of the Aztec diamond, extending the work of Borodin and Berggren [BB23]. It would be interesting to use the spectral transform for electrical networks, which I studied in [Geo24a], to establish stronger limit shape results for groves.

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