RESEARCH STATEMENT

TERRENCE GEORGE

My research lies at the intersection of combinatorics, probability, and algebraic geometry. I have explored combinatorial models in statistical mechanics—specifically the dimer model, electrical networks, and the Ising model—and their connections to discrete geometry, integrable systems, algebraic geometry, etc.

The **dimer model** was originally studied in statistical physics as a model for the adsorption of diatomic molecules on crystal surfaces. It can also be interpreted as a model for a melting crystal. Mathematically, a **dimer cover** (or **perfect matching**) of a graph is a subset of the edges that use every vertex exactly once (Figure 1). The enumeration of perfect matchings is a classical problem in graph theory and is known to be computationally hard. A celebrated result of Kasteleyn [Kas63] and of Temperley and Fischer [TF61] says that, for planar bipartite graphs, the number of dimer covers of a can be computed in polynomial time as the determinant of a version of the adjacency matrix of the graph, called the **Kasteleyn matrix**. Underpinning the success of the dimer model is this determinantal structure, which effectively reduces probabilistic computations for the dimer model to linear algebra.

Beyond statistical mechanics, recent research has uncovered surprising and profound connections between the determinantal structure of the dimer model and various fields, including total positivity in combinatorics, integrable aspects of discrete differential geometry, discrete complex analysis and graph embeddings, and Harnack curves and Beauville integrable systems in algebraic geometry. Through my work, I aim to contribute to a deeper understanding of these relationships and the broader implications of the dimer model across these various fields, with a particular emphasis on integrating geometric and algebraic methods.

1. Limit shapes

A fascinating aspect of dimer models is the emergence of **limit shapes**, a manifestation of the law of large numbers, where a random dimer cover of a large graph concentrates around a deterministic shape. The first result of this kind was the celebrated Arctic Circle Theorem for domino tilings [JPS98]. These limit shapes can be interpreted as modeling melting crystals in

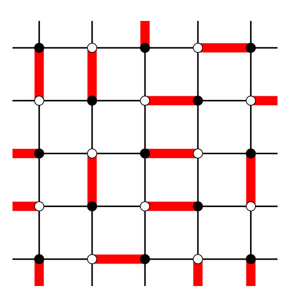
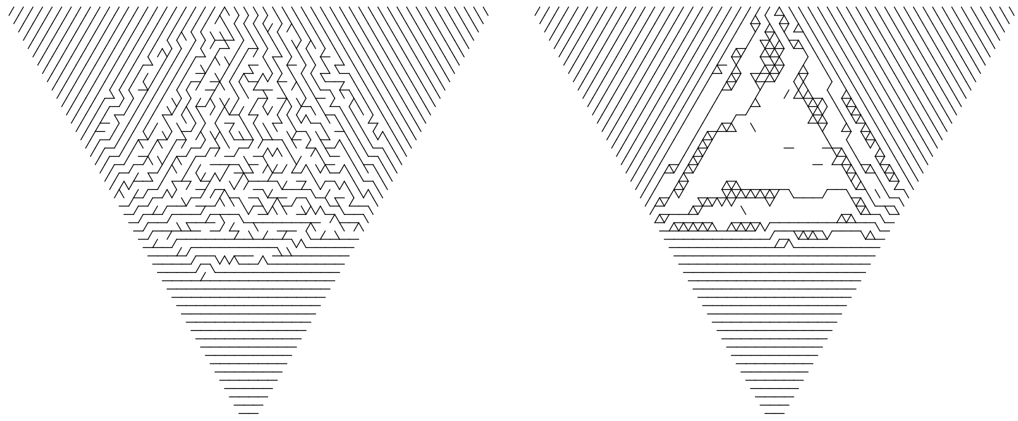


Figure 1. A dimer cover of a portion of the square lattice graph that models the surface of a crystal with two types of molecules corresponding to black and white vertices.



(a) Grove (b) FK percolation

Figure 2. Simulations of random groves and percolation configurations on the triangular lattice showing the emergence of limit shapes. In each picture, there is a "liquid" region in the middle outside which the random object appears frozen or "solid."

equilibrium and have distinct macroscopic regions corresponding to solid and liquid phases. A common theme is that the results for the dimer model have parallels in other statistical mechanical models, such as electrical networks and Ising models. In electrical networks, the equivalent of dimer covers are spanning-tree-like objects called **groves**. In [Geo21], I used the integrable structure of the model to extend the Arctic Circle Theorem for groves due to Petersen and Speyer [PS05] (Figure 2(a)). While this technique only identifies the boundary between different phases and not the full limit shape, it shows that groves display similarly rich limit shape behavior as dimers and deserve further study using more powerful techniques such as variational methods. With Russkikh, Kenyon, and Vu, I am investigating limit shapes in Fortuin–Kasteleyn (FK) percolation models, a one-parameter family of statistical-mechanical models generalizing groves (Figure 2(b)).

2. Total positivity

Many of the connections of the dimer model to other areas of mathematics involve moduli spaces of weighted bipartite graphs on surfaces, which possess a rich underlying structure. These spaces are instances of Fock and Goncharov's **cluster varieties** [FG09], the geometric analogs of Fomin and Zelevinsky's cluster algebras [FZ02].

The **Grassmannian**, the space parameterizing all subspaces of a fixed dimension of a vector space, is a classical object in algebraic geometry. An explicit way to present an element of the Grassmannian is to specify a rectangular matrix whose rows span the subspace. The maximal minors of the matrix provide a set of homogeneous coordinates called **Plücker coordinates**. A totally nonnegative matrix is a matrix whose minors are all nonnegative. Generalizing this notion, Postnikov [Pos06] defined the **totally nonnegative Grassmannian** to be the set of elements of the Grassmannian with all Plücker coordinates nonnegative. Roughly, the Kasteleyn matrix of a dimer model on a disk is a matrix representative of an element of the totally nonnegative Grassmannian. Postnikov showed that this construction, called **boundary measurement**, is a bijection between the space of weighted bipartite

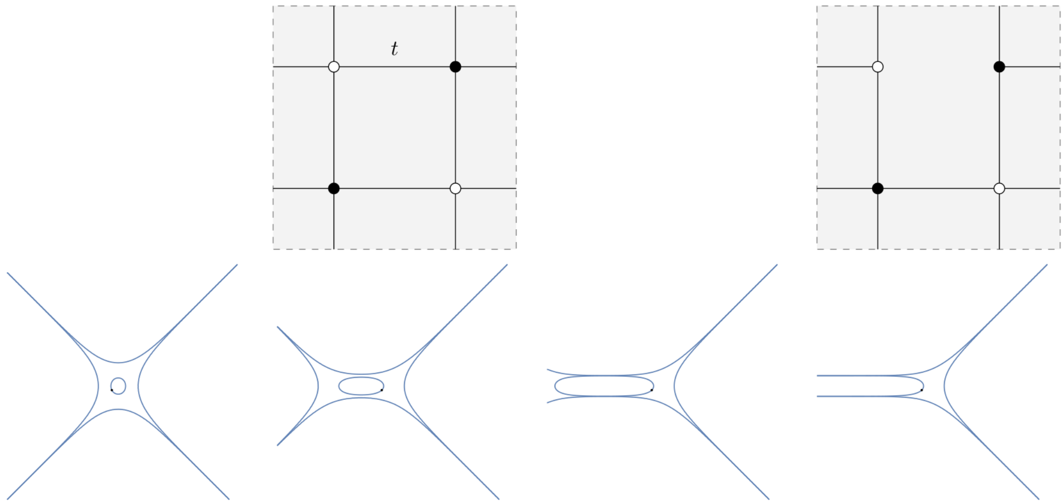


FIGURE 3. Degeneration of a weighted bipartite graph with weights as $t \to 0$, and the corresponding degeneration of the (amoeba of the) Harnack curve and line bundle (drawn as a divisor).

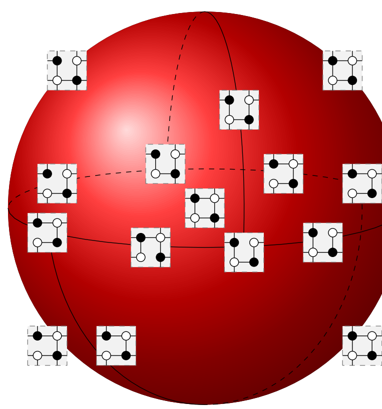


FIGURE 4. An example of a stratified "totally nonnegative" space parameterizing degenerations of weighted bipartite graphs on a torus/Harnack curves and line bundles. Each cell is labeled by an equivalence class of bipartite graphs, a representative of which is shown.

graphs and the totally nonnegative Grassmannian. Moreover, degenerations of weighted bipartite graphs induce a beautiful stratification of the totally nonnegative Grassmannian into **positroid** cells. Lam [Lam18] showed that boundary measurements of electrical networks form a stratified subspace of the totally nonnegative Grassmannian. With Chepuri and Speyer [CGS21], we further showed that this subspace is a totally nonnegative Lagrangian Grassmannian. Together with the work of Galashin and Pylyavskyy on the Ising model [GP20], this completes a compelling classification of totally nonnegative Grassmannians associated with statistical-mechanical models in the disk. In [Geo24c], I studied the problem of explicitly reconstructing an electrical network from its boundary measurements. This is a classical problem which has been studied by many authors over the years. I found a new, and in a sense, natural solution using ideas from cluster algebras.

3. Total positivity on a torus

Aside from the disk, the best-understood dimer models are those on a torus. Traditionally, these models were studied in statistical mechanics because they help analyze the behavior of planar graphs as the graphs become larger. A way to take the limit that avoids boundary issues is to take larger and larger covers of a graph on a torus. In this setting, it is natural to consider the action of the Kasteleyn matrix on translation eigenspaces. This modified Kasteleyn matrix contains two variables z, w (translation eigenvalues), so its determinant is a polynomial in z, w and therefore defines an algebraic curve called the **spectral curve**. Kenyon, Okounkov, and Sheffield [KOS06] proved a surprising and deep connection between the dimer model and real algebraic geometry. They showed that the spectral curve is a real algebraic curve with a special topology called a **Harnack curve** and used this fact to obtain a universal classification of the phases of the dimer model. Later, Kenyon and Okounkov [KO06] showed that there is a bijection, called **spectral transform**, between the space of weighted bipartite graphs on the torus and the moduli space of Harnack curves along with specific line bundles on them, or equivalently, points in the Jacobian variety. The spectral transform is the torus-analog of Postnikov's boundary measurement on the disk and, jointly with Pavel Galashin, one of my long-term research goals is to understand degenerations of weighted bipartite graphs on the torus (Figure 3) and the induced positroid-like stratification on the moduli space of Harnack curves with line bundles. As a first step, in [GG24], we study the combinatorics of the stratification and determine the analog of positroids, which is already considerably more complicated and subtle than in Postnikov's setting (Figure 4). The next step would be to geometry of the strata, which we presently only understand in examples of small genus.

4. The inverse spectral transform

Cluster varieties are equipped with a canonical Poisson structure [GSV10] compatible with local transformations called cluster mutations. The dimer model has a local transformation called the **spider move**, known from the beginnings of cluster algebras as an example of a cluster mutation. Upgrading these observations, Goncharov and Kenyon [GK13] constructed, for each convex polygon in the plane with integer vertices, a Hamiltonian integrable system called a **cluster integrable system** from the dimer model on a torus. It was later shown in [FM16] that many well-known integrable systems, such as the Toda lattice and the pentagram map, are special cases of cluster integrable systems. The **spectral transform** of Kenyon and Okounkov provides action-angle coordinates for the integrable system, a set of coordinates in which the system's time evolution is linear. Fock [Foc15], using the theory of finite-gap solutions of integrable systems, explicitly constructed the inverse of the spectral transform and showed that the spider move is equivalent to Fay's celebrated trisecant identity for theta functions on Riemann surfaces. With Goncharov and Kenyon [GGK23], I gave an alternative construction of the inverse spectral transform that only involves rational functions (as opposed to the transcendental theta functions) and is hence amenable to symbolic computation.

In [Geo24a], I studied the generalization of cluster integrable systems to electrical networks. I showed that the spectral curve is a Harnack curve with an involution. Whenever a curve carries an involution, it determines a linear subvariety of the Jacobian variety called the Prym variety. I explicitly constructed the inverse of the spectral transform using Prym theta functions. Further, I showed that the local move for electrical networks, the star-triangle transformation, is equivalent to Fay's quadrisecant identity for Prym theta functions. In [Geo24b], I investigated the Ising model on the torus and showed, through non-explicit methods, that the spectral transform forms a bijection with a space of symmetric Harnack curves and Prym varieties. These papers reveal that these models possess many, if not all, of the integrable structures present in the dimer model, and I believe there is still much to be explored in this area. An interesting open problem is constructing an explicit inverse map using theta functions and finding the theta function identity describing the Ising star-triangle transformation. Another significant problem is finding Poisson structures compatible with electrical and Ising star-triangle transformations and proving Arnold–Louville's integrability.

Associated with a convex polygon having integer coordinates in the plane is another integrable system from algebraic geometry: the **Beauville integrable system** [Bea91, Bea90] on the corresponding projective toric surface. Goncharov and Kenyon [GK13] conjectured that the spectral transform is an isomorphism between integrable systems. After the initial strategy of Goncharov and Kenyon to prove this conjecture through combinatorial methods was unsuccessful, Inchiostro and I proved it in [GI22] using homological algebra machinery for the dimer model on the hexagonal lattice, i.e., in the case where the toric surface is the projective plane. Establishing the equivalence of cluster integrable systems with Beauville integrable systems, in general, would be very interesting. I also believe that these powerful, though abstract, algebraic techniques could have broader applications than what we have explored so far.

5. Discrete integrable systems

We obtain discrete dynamical systems by applying sequences of spider moves on weighted bipartite graphs, known as **cluster modular transformations** as defined in [FG09]. An example is the pentagram map [OST10], a cluster modular transformation on a specific square lattice. In addition to integrable systems in geometry, cluster modular transformations also emerge from the combinatorial and probabilistic study of the dimer model. These transformations have been used to compute partition functions, perform exact sampling via shuffling algorithms, and prove limit shape results (see, e.g., [EKLP92, JPS98, PS05, KP16]). Fock and Marshakov [FM16] investigated the group of cluster modular transformations and conjectured its isomorphism type, a conjecture that Inchiostro and I confirmed in [GI24].

Recently, several integrable systems have been discovered in discrete differential geometry (see, e.g., [ILP16, AFIT22, Izo23]) that belong to the class of cluster integrable systems. However, their discrete dynamics are governed not only by sequences of spider moves but also by an additional non-local transformation known as the geometric R-matrix, first introduced by Inoue, Lam, and Pylyavskyy [ILP16]. The group of discrete dynamics that arises in this context is termed the generalized cluster modular group, which Ramassamy and I computed in [GR23], extending the results from [GI24].

Together with Affolter and Ramassamy, I developed a general framework in [AGR21] to prove the equivalence of these integrable systems with cluster integrable systems. Unlike previous approaches that relied on algebraic methods to compare coordinates, our work introduces a novel geometric perspective to make this identification. This approach, along with [AGR] and [AGPR24], provides a unified understanding of the integrable aspects of discrete differential geometry, dimer models, and cluster algebras.

6. Other short and medium-term projects

I am excited about working with students on research projects. Below, I outline some potential projects aligning with my research interests and knowledge that would be well-suited for PhD students or highly motivated Masters students.

6.1. **The twist map for electrical networks/Ising models.** Cluster varieties are part of an ensemble $\mathcal{A} \to \mathcal{X}$, comprising two varieties, $\mathcal{A}$ and $\mathcal{X}$, with a canonical map between them. In the case of the dimer model on a disk, both $\mathcal{A}$ and $\mathcal{X}$ are subvarieties of the Grassmannian, called open positroid varieties [KLS13]. The canonical map is an automorphism between these varieties, called the *twist* [MS16, MS17]. Although electrical networks and Ising models are part of similar cluster-like ensembles [GK13, KP16], where the $\mathcal{X}$ varieties are relatively well understood [BGKT23, GP20, Lam18, CGS21], the $\mathcal{A}$ varieties and the twist require further investigation. I worked out the simplest case in [Geo24c].

6.2. **Positive geometry.** An exciting development at the intersection of combinatorics and algebraic geometry, inspired by studying scattering amplitudes in physics, is the emerging field of Positive Geometry [AHBL17]. Lam [Lam16] showed that positroid varieties are instances of positive geometries. It would be interesting to prove that electrical networks and Ising models are also examples of positive geometries. Doing this requires new approaches to investigate the geometric properties of these spaces, as the representation-theoretic methods used for positroid varieties appear to be inadequate.

6.3. **Electrical matrix Schubert varieties.** Matrix Schubert varieties, introduced by Fulton [Ful92], are determinantal varieties whose Gröbner degenerations were shown by Knutson and Miller [KM05] to provide a geometric explanation for the positive expansion of Schubert polynomials using pipe dreams. Snider [Sni10] extended this to open patches in positroid varieties and affine pipe dreams. Together with David Speyer, we computed several examples of analogous degenerations of response matrices in electrical networks characterized by objects resembling pipe dreams, which would be very interesting to investigate further.

6.4. **Limit shapes for cube groves using the spectral transform.** Recently, Boutillier and de Tilière [BdT24] applied Fock's inverse spectral transform [Foc15] for the dimer model to investigate domino tilings of the Aztec diamond, extending the work of Borodin and Berggren [BB23]. It would be interesting to use the spectral transform for electrical networks, which I studied in [Geo24a], to establish stronger limit shape results for groves.

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