# The inverse spectral map for dimers 

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#### Abstract

In 2015, Vladimir Fock proved that the spectral transform, associating to an element of a dimer cluster integrable system its spectral data, is birational by constructing an inverse map using theta functions on Jacobians of spectral curves. We provide an alternate construction of the inverse map that involves only rational functions in the spectral data.


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## 1 Introduction

The planar dimer model is a classical statistical mechanics model, involving the study of the set of dimer covers (perfect matchings) of a planar, edge-weighted graph. In the 1960s, Kasteleyn Kas61, Kas63] and Temperley and Fisher TF61] showed how to compute the (weighted) number of dimer covers of planar graphs using the determinant of a signed adjacency matrix now known as the Kasteleyn matrix.

In mathematics the dimer model was popularized with the papers EKLP92a, EKLP92b] on the "Aztec diamond" and later with results on the local statistics Ken97, conformal invariance Ken00, and limit shapes [KP01, connections with algebraic geometry KOS06, KO06, cluster varieties and integrability GK13, and string theory HK05.

While the dimer model can be considered from a purely combinatorial point of view, it also has a rich integrable structure, first described in GK13. The integrable structure on dimers on graphs on the torus was found to generalize many well-known integrable systems, see for example [FM16] and AGR21. What is especially important is that the related integrable system is of cluster nature, and this allows one to immediately quantize it, getting a quantum integrable system.

From the point of view of classical mechanics, associated to the dimer model on a bipartite graph on a torus (or equivalently a periodic bipartite planar graph) is a Poisson variety with a Hamiltonian integrable system. Underlying this system is an algebraic curve $\mathcal{C}=\{P(z, w)=0\}$ (called the spectral curve) and a divisor on this curve-essentially a set of $g$ distinct points $\left\{\left(p_{1}, q_{1}\right), \ldots,\left(p_{g}, q_{g}\right)\right\}$ on $\mathcal{C}$. This is the spectral data associated to the model. It was shown in KO06 that the map from the weighted graph to the spectral data was bijective, from the space of "face weights" (see below) to the moduli space of genus- $g$ curves and effective degree- $g$ divisors on the open spectral curve $\mathcal{C}^{\circ}$. Subsequently Fock Foc15] constructed the inverse spectral map (from the spectral data to the face weights), describing it in terms of theta functions over the spectral curve. The special case of genus 0 was described earlier in Ken02 KO06] and an explicit construction in the case of genus 1 was more recently given in BCdT20. Positivity of Fock's inverse map was studied in BCdT21].

In the current paper, we show that the inverse map can be given an explicit rational expression in terms of the divisor points $\left(p_{i}, q_{i}\right) \in \mathcal{C}^{\circ}$ and the points of $\mathcal{C}$ at toric infinity. An exact statement is given in Theorem 3.10 below.

While Fock's construction is very natural and interacts nicely with positivity, it involves theta functions. Our construction gives the inverse map as ratios of certain determinants in the spectral data and can be explicitly computed using computer algebra. We briefly describe our construction now. The spectral data is defined via a matrix $K=K(z, w)$ called the Kasteleyn matrix, whose rows are indexed by white vertices, columns by black vertices, and whose entries are Laurent polynomials in $z$ and $w$. Let us consider the adjugate matrix of $K$ :

$$
Q=Q(z, w)=K^{-1} \operatorname{det} K .
$$

The matrix $Q$ is important when studying the probabilistic aspects of the dimer model (on the lift of the graph on the torus to the plane): the edge occupation variables form a determinantal process whose kernel is given by the Fourier coefficients of $Q / P$, as discussed in KOS06. In the present work, we have a different use for $Q$ : finding (a column of) the matrix $Q$ from the spectral data allows us to reconstruct the face weights and thereby invert the spectral transform.

The points $\left(p_{i}, q_{i}\right) \in \mathcal{C}$ are defined to be the points where a column of $Q$, corresponding to a fixed white vertex $\mathbf{w}$, vanishes. We show that entries in the w-column of $Q$, which are Laurent polynomials, can be reconstructed from the spectral data by solving a linear system of equations. Some of the linear equations are easy to describe: for any black vertex b, we have $Q_{\mathrm{bw}}\left(p_{i}, q_{i}\right)=0$ for $i=1, \ldots, g$, which are $g$ linear equations in the coefficients of the Laurent polynomial $Q_{\mathrm{bw}}$. However, these equations are usually not sufficient to determine the coefficients of $Q_{\mathrm{bw}}$. We find additional equations from the vanishing of $Q_{\mathrm{bw}}$ at certain points at infinity of the spectral curve $\mathcal{C}$, and show that these equations determine $Q_{\mathrm{bw}}$ uniquely, up to a non-zero constant. We then give a procedure to reconstruct the weights from the w-column of $Q$.

A key construction in our approach is the extension of the Kasteleyn matrix $K$ to a map of vector bundles on a toric stack, for which we make crucial use of the classification of line bundles on toric stacks and the computation of their cohomology developed in BH09. Toric stacks already appear implicitly in the context of the spectral transform in KO06 and explicitly TWZ19.

The article is organized as follows. In Section 2 we review the dimer cluster integrable system and the spectral transform. In Section 3, we state Theorem 3.1, which is our main result, and describe the reconstruction procedure. We work out two detailed examples in Section 4 Sections 5. 6 and 7 contain proofs of our results. In Appendix A, we review results from toric geometry. In Appendix B we provide explicit combinatorial descriptions for some of our constructions. These are useful for computations.

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## 2 Background

For further information about the material in this section see GK13.

### 2.1 Dimer models

Let $\Gamma$ be a bipartite graph on the torus $\mathbb{T} \cong S^{1} \times S^{1}$ such that the connected components of the complement of $\Gamma$-the faces-are contractible. We denote by $B(\Gamma)$ and $W(\Gamma)$ the black and white vertices of $\Gamma$, by $V(\Gamma)$ the vertices, and by $E(\Gamma)$ the edges of $\Gamma$. When the graph is clear from context, we will usually abbreviate these to $B, W, V$ and $E$.

A dimer model on the torus is a pair $(\Gamma,[w t])$, where $\Gamma$ is a bipartite graph on the torus as above and $[\omega t] \in H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$(Here and throughout the paper, $\mathbb{C}^{\times}$denotes the group of nonzero complex numbers under multiplication). For a loop $L$ and a cohomology class $[w t]$, we denote by $[w t]([L])$ the pairing between the cohomology and the homology. We orient edges from their black vertex to their white vertex. The cohomology class $[w t]$ can be represented by a cocycle $w t$ which, using this orientation, can be identified with a $\mathbb{C}^{\times}$-valued function on the edges of $\Gamma$ called an edge weight.

The edge weight is well-defined modulo multiplication by coboundaries, which are functions on edges $e=\mathrm{bw}$ given by $f(\mathrm{w}) f(\mathrm{~b})^{-1}$ for functions $f: V(\Gamma) \rightarrow \mathbb{C}^{\times}$. Note that the weight of a loop is not the product of its edge weights, but the "alternating product" of its edge weights: edges oriented against the orientation of the loop are multiplied with exponent -1 .

A dimer cover or perfect matching m of $\Gamma$ is a subset of $E(\Gamma)$ such that each vertex of $\Gamma$ is incident to exactly one edge in m . Let $\mathcal{M}$ denote the set of dimer covers of $\Gamma$. If we fix a reference dimer cover $\mathrm{m}_{0}$, we get a function

$$
\begin{aligned}
\pi_{\mathrm{m}_{0}}: & \mathcal{M} \\
& \rightarrow H_{1}(\mathbb{T}, \mathbb{Z}) \\
& \mathrm{m}
\end{aligned}>\left[\mathrm{m}-\mathrm{m}_{0}\right] .
$$

Here $\mathrm{m}-\mathrm{m}_{0}$ is the 1 -chain which assigns 1 to (oriented) edges of m and -1 to (oriented) edges of $m_{0}$, so $m-m_{0}$ is a union of oriented cycles and doubled edges, whose homology class is [ $\mathrm{m}-\mathrm{m}_{0}$ ].

The Newton polygon of $\Gamma$ is the polygon

$$
N(\Gamma):=\text { Convex-hull }\left(\pi_{\mathrm{m}_{0}}(\mathcal{M})\right) \subset H_{1}(\mathbb{T}, \mathbb{R})
$$

defined modulo translation by $H_{1}(\mathbb{T}, \mathbb{Z})$. Changing the reference dimer cover from $\mathrm{m}_{0}$ to $\mathrm{m}_{0}^{\prime}$ results in a translation of the polygon by $\left[\mathrm{m}_{0}-\mathrm{m}_{0}^{\prime}\right]$, so the Newton polygon does not depend on the choice.

We assume that $\Gamma$ is such that $N(\Gamma)$ has interior. This is a nondegeneracy condition on $\Gamma$. (When $N$ has empty interior, the graph $\Gamma$ is equivalent under certain elementary transformations to a graph whose lift to $\mathbb{R}^{2}$ is disconnected, that is, has noncontractible faces; such a graph breaks into essentially one-dimensional components, and there is no integrable system.)

### 2.2 Zig-zag paths and the Newton polygon

A zig-zag path in $\Gamma$ is a closed path that turns maximally right at each black vertex and maximally left at each white vertex. The medial graph of $\Gamma$ is the graph $\Gamma^{\times}$that has a vertex $v_{e}$ at the midpoint of each edge $e$ of $\Gamma$ and an edge between $v_{e}$ and $v_{e^{\prime}}$ whenever $e$ and $e^{\prime}$ occur consecutively around a face of $\Gamma$. Note that by construction, each vertex of $\Gamma^{\times}$has degree 4 . A zig-zag path in $\Gamma$ corresponds to a cycle in $\Gamma^{\times}$that goes straight through each degree four vertex, i.e., at every vertex, the outgoing edge of the cycle is the one that is opposite the incoming one (see Figure (2). Hereafter, when we say zig-zag path, we mean the corresponding cycle in the medial graph.

Let $\widetilde{\Gamma}$ be the biperiodic graph on the plane given by the lift of $\Gamma$ to the universal cover of $\mathbb{T}$. The bipartite graph $\Gamma$ is said to be minimal if the lift of any zig-zag path does not self-intersect, and lifts of any two zig-zag paths do not have "parallel bigons", where by parallel bigon we mean two


Figure 1: The fundamental rectangle $R$, along with the cycles $\gamma_{z}, \gamma_{w}$.

$\Gamma$

$\Gamma^{\times}$

Figure 2: A zig-zag path in a graph $\Gamma$ and the corresponding cycle in the medial graph $\Gamma^{\times}$.
consecutive intersections where both paths are oriented in the same direction from one to the next. For a minimal bipartite graph $\Gamma$ on the torus, the Newton polygon has an alternative description in terms of the zig-zag paths of $\Gamma$. Namely, since $\Gamma$ is embedded in $\mathbb{T}$, each zig-zag path $\alpha$ has a non-zero homology class $[\alpha] \in H_{1}(\mathbb{T}, \mathbb{Z})$. The polygon $N(\Gamma)$ is the unique convex integral polygon defined modulo translation in $H_{1}(\mathbb{T}, \mathbb{Z})$ whose integral primitive edge vectors in counterclockwise order around $N$ are given by the vectors $[\alpha]$ for all zig-zag paths $\alpha$.

Example 2.1. Consider the fundamental domain for the square lattice shown in Figure 1 and let $\gamma_{z}, \gamma_{w}$ be cycles generating $H_{1}(\mathbb{T}, \mathbb{Z})$ as shown there. We will write homology classes in $H_{1}(\mathbb{T}, \mathbb{Z})$ in the basis $\left(\gamma_{z}, \gamma_{w}\right)$. There are four zig-zag paths labeled $\alpha, \beta, \gamma$ and $\delta$ with homology classes $(-1,1),(-1,-1),(1,-1)$ and $(1,1)$ respectively (Figure 3), and therefore the Newton polygon is

$$
\text { Convex-hull }\{(1,0),(0,1),(-1,0),(0,-1)\} .
$$

### 2.3 The cluster variety assigned to a Newton polygon

For a convex integral polygon $N \subset H_{1}(\mathbb{T}, \mathbb{R})$ defined modulo translation, consider the family of minimal bipartite graphs $\Gamma$ with Newton polygon $N(\Gamma)=N$. Any two graphs $\Gamma_{1}, \Gamma_{2}$ in the family are related by certain elementary transformations; see Figure 4. An elementary transformation $\Gamma_{1} \rightarrow \Gamma_{2}$ gives rise to a birational map $H^{1}\left(\Gamma_{1}, \mathbb{C}^{\times}\right) \rightarrow H^{1}\left(\Gamma_{2}, \mathbb{C}^{\times}\right)$. Gluing the tori $H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$by these maps, we obtain a space $\mathcal{X}_{N}$, called the dimer cluster Poisson variety. It carries a canonical


Figure 3: Zig-zag paths and Newton polygon for the bipartite graph in Figure 1


Figure 4: The elementary transformations.

Poisson structure. The Poisson center is generated by the loop weights of the zig-zag paths. The space $\mathcal{X}_{N}$ is the phase space of the cluster integrable system. See details in GK13.

### 2.4 Some notation

Let $\Sigma$ denote the normal fan of $N$ (see Section A.2 and Figures 7 and 10) so that the set of rays $\Sigma(1)=\{\rho\}$ of $\Sigma$ is in bijection with the set of edges of $N$. We denote the edge of $N$ whose inward normal is directed along the ray $\rho$ by $E_{\rho}$, and the primitive vector along $\rho$ by $u_{\rho}$.

Let $\mathrm{M}:=H_{1}(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^{2}$ and $\mathrm{M}^{\vee}:=\operatorname{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z}) \cong \mathbb{Z}^{2}$ be dual lattices and let $\langle *, *\rangle: \mathrm{M} \times \mathrm{M}^{\vee} \rightarrow$ $\mathbb{Z}$ denote the duality pairing. Let us consider the algebraic torus with lattice of characters M :

$$
\mathrm{T}:=\operatorname{Hom}_{\mathbb{Z}}\left(\mathrm{M}, \mathbb{C}^{\times}\right) \cong\left(\mathbb{C}^{\times}\right)^{2}
$$

Let $\mathrm{M}_{\mathbb{R}}\left(\right.$ resp. $\left.\mathrm{M}_{\mathbb{R}}^{\vee}\right)$ denote $\mathrm{M} \otimes_{\mathbb{Z}} \mathbb{R}\left(\right.$ resp. $\left.\mathrm{M}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}\right)$, so that $N \subset \mathrm{M}_{\mathbb{R}}$ and $\Sigma \subset \mathrm{M}_{\mathbb{R}}^{\vee}$.
An elementary transformation $\Gamma_{1} \rightarrow \Gamma_{2}$ induces a canonical bijection between zig-zag paths in $\Gamma_{1}$ and zig-zag paths in $\Gamma_{2}$. Therefore, the set of zig-zag paths is canonically associated with $N$. We denote the set of zig-zag paths by $Z$, and for an edge $E_{\rho}$ of $N$, we denote by $Z_{\rho}$ the set of zig-zag paths $\alpha$ such that the primitive vector $[\alpha]$ is contained in $E_{\rho}$.

### 2.5 The Kasteleyn matrix

Let $R$ be a fundamental rectangle for $\mathbb{T}$, so that $\mathbb{T}$ is obtained by gluing together opposite sides of $R$. Let $\gamma_{z}, \gamma_{w}$ be the oriented sides of $R$ generating $H_{1}(\mathbb{T}, \mathbb{Z})$, as shown in Figure 1 Let $z$ (resp. $w)$ denote the character $\chi^{\gamma_{w}}$ (resp. $\chi^{\gamma_{z}}$ ), so the coordinate ring of T is $\mathbb{C}\left[z^{ \pm 1}, w^{ \pm 1}\right]$.

Let $(*, *)_{\mathbb{T}}$ be the intersection pairing on $H_{1}(\mathbb{T}, \mathbb{Z})$. For $z, w \in \mathbb{C}^{\times}$we multiply edge weights on edges crossing $\gamma_{z}$ by $z^{ \pm 1}$ and those crossing $\gamma_{w}$ by $w^{ \pm 1}$, with the sign determined by the orientation. Precisely, we multiply by

$$
\begin{equation*}
\phi(e):=z^{\left(e, \gamma_{w}\right)_{\mathbb{T}}} w^{\left(e,-\gamma_{z}\right)_{\mathbb{T}}} \tag{1}
\end{equation*}
$$

Here $(e, *)_{\mathbb{T}}:=\left(l_{e}, *\right)_{\mathbb{T}}$ is the intersection index with the oriented loop $l_{e}$ obtained by concatenating $e=\mathrm{bw}$ with an oriented path contained in $R$ from w to b .

A Kasteleyn sign is a cohomology class $[\epsilon] \in H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$such that for any loop $L$ in $\Gamma,[\epsilon]([L])$ is -1 (resp., 1$)$ if the number of edges in $L$ is $0 \bmod 4($ resp., $2 \bmod 4)$. Given edge weights $w t$ and $\epsilon$ representing $[w t]$ and $[\epsilon]$ respectively, one defines the Kasteleyn matrix $K=K(z, w)$, whose columns and rows are parameterized by $\mathrm{b} \in B$ and $\mathrm{w} \in W$ respectively:

$$
\begin{equation*}
K_{\mathrm{w}, \mathrm{~b}}=\sum_{e \in E \text { incident to } \mathrm{b}, \mathrm{w}} w t(e) \epsilon(e) \phi(e) \tag{2}
\end{equation*}
$$

It describes a map of free $\mathbb{C}\left[z^{ \pm 1}, w^{ \pm 1}\right]$-modules, called the Kasteleyn operator:

$$
\begin{align*}
K: & \mathbb{C}\left[z^{ \pm 1}, w^{ \pm 1}\right]^{B} \rightarrow \mathbb{C}\left[z^{ \pm 1}, w^{ \pm 1}\right]^{W}  \tag{3}\\
& \delta_{\mathrm{b}} \longmapsto \sum_{\mathrm{w} \in W} K_{\mathrm{w}, \mathrm{~b}} \delta_{\mathrm{w}} \tag{4}
\end{align*}
$$

Theorem 2.2 (Kasteleyn 1963, Kas63]). Fix a dimer cover $\mathrm{m}_{0}$, and let $\phi\left(\mathrm{m}_{0}\right)=\prod_{e \in \mathrm{~m}_{0}} \phi(e)$. Then,

$$
\frac{1}{w t\left(\mathrm{~m}_{0}\right) \epsilon\left(\mathrm{m}_{0}\right) \phi\left(\mathrm{m}_{0}\right)} \operatorname{det} K=\sum_{\mathrm{m} \in \mathcal{M}} \operatorname{sign}\left(\left[\mathrm{~m}-\mathrm{m}_{0}\right]\right)[w t]\left(\left[\mathrm{m}-\mathrm{m}_{0}\right]\right) \chi^{\left[\mathrm{m}-\mathrm{m}_{0}\right]}
$$



Figure 5: Shown on the left is a labeling of vertices and faces of $\Gamma$, and two cycles $a$ (red) and $b$ (green) in $\Gamma$ that generate $H_{1}(\mathbb{T}, \mathbb{Z})$. Shown on the right is a cocycle representing [ $w t$ ], along with $\epsilon$ and $\phi$. The signs are due to $\epsilon$, the $z, w$ due to $\phi$, and other weights are $w t$.
where $\operatorname{sign}\left(\left[\mathrm{m}-\mathrm{m}_{0}\right]\right) \in\{ \pm 1\}$ is a sign that depends only on the homology class $\left[\mathrm{m}-\mathrm{m}_{0}\right]$ and $[\epsilon]$.
The characteristic polynomial is the Laurent polynomial

$$
P(z, w):=\frac{1}{w t\left(\mathrm{~m}_{0}\right) \epsilon\left(\mathrm{m}_{0}\right) \phi\left(\mathrm{m}_{0}\right)} \operatorname{det} K
$$

Its vanishing locus $\mathcal{C}^{\circ}:=\{P(z, w)=0\} \subset\left(\mathbb{C}^{\times}\right)^{2}$ is called the (open part of the) spectral curve. Theorem 2.2 implies that $N$ is the Newton polygon of $P(z, w)$. Although the definition of the Kasteleyn matrix uses cocycles representing the cohomology classes $w t$ and $\epsilon$, the spectral curve does not depend on these choices.

Example 2.3. Let $a$ and $b$ be the two cycles in $\Gamma$ shown on the left of Figure 5 whose projections to $\mathbb{T}$ generate $H_{1}(\mathbb{T}, \mathbb{Z})$. Let $[w t] \in H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$and let $A:=[w t]([a]), B:=[w t]([b])$. For $i=1,2,3$, let $X_{i}$ denote the $[w t]\left(\left[\partial f_{i}\right]\right)$, where $\partial f_{i}$ denotes the boundary of the face $f_{i}$ (the weight of the fourth face is determined by the fact that the product of all face weights is 1$)$. Then $\left(X_{1}, X_{2}, X_{3}, A, B\right)$ generate the coordinate ring of $H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$. A cocycle representing $[w t]$ is shown on the right of Figure [5] along with $\epsilon$ and $\phi$. The Kasteleyn matrix and the spectral curve are:

$$
\begin{gather*}
K=\left(\begin{array}{cc}
\mathrm{b}_{1} & \mathrm{~b}_{2} \\
1-A z & 1-\frac{X_{1} X_{3}}{B w} \\
-1+B w & X_{1}-\frac{1}{A X_{2} z}
\end{array}\right) \mathrm{w}_{1} \\
\mathrm{w}_{2}
\end{gather*} \begin{aligned}
& P(z, w)=\left(1+X_{1}+\frac{1}{X_{2}}+X_{1} X_{3}\right)-B w-\frac{X_{1} X_{3}}{B w}-\frac{1}{A X_{2} z}-A X_{1} z . \tag{5}
\end{aligned}
$$

### 2.6 The toric surface assigned to a Newton polygon

In this section, we collect some notation regarding toric varieties, and refer the reader to the Appendices A.1 and A.2 for more details. A convex integral polygon $N \subset \mathrm{M}_{\mathbb{R}}$ determines a
compactification $X_{N}$ of the complex torus T called a toric surface, and a divisor $D_{N}$ supported on the boundary $X_{N}-\mathrm{T}$, so that Laurent polynomials with Newton polygon $N$ extend naturally to sections of the coherent sheaf $\mathcal{O}_{X_{N}}\left(D_{N}\right)$ (for background on the coherent sheaf associated to a divisor, see for example [CLS11, Chapter 4]).

Denote by $\left|D_{N}\right|$ the projective space of non-zero global sections of the coherent sheaf $\mathcal{O}_{X_{N}}\left(D_{N}\right)$, considered modulo a multiplicative constant. Assigning to a section its vanishing locus, we see elements of $\left|D_{N}\right|$ as curves in $X_{N}$ whose restrictions to T are defined by Laurent polynomials with Newton polygon contained in $N$.

The genus $g$ of the generic curve in $\left|D_{N}\right|$ is equal to the number of interior lattice points in $N$. Recall that the edges $\left\{E_{\rho}\right\}$ of $N$ are in bijection with the rays $\{\rho\}$ of $\Sigma$. Each edge $E_{\rho}$ of $N$ determines a projective line $D_{\rho}$ which we call a line at infinity of $X_{N}$, and

$$
X_{N}-\mathrm{T}=\bigcup_{\rho \in \Sigma(1)} D_{\rho}
$$

The divisor $D_{N}$ is given by

$$
\begin{equation*}
D_{N}=\sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho} \tag{6}
\end{equation*}
$$

where $a_{\rho} \in \mathbb{Z}$ are such that

$$
\begin{equation*}
N=\bigcap_{\rho \in \Sigma(1)}\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle \geq-a_{\rho}\right\} \tag{7}
\end{equation*}
$$

The lines $D_{\rho}$ intersect according to the combinatorics of $N$ : precisely, for $\rho_{1}, \rho_{2} \in \Sigma(1)$, the intersection $D_{\rho_{1}} \cap D_{\rho_{2}}$ is empty if $E_{\rho_{1}} \cap E_{\rho_{2}}$ is empty and a point if $E_{\rho_{1}} \cap E_{\rho_{2}}$ is a vertex of $N$. The intersection index of a generic curve in $\left|D_{N}\right|$ with the line $D_{\rho}$ is equal to the number $\left|E_{\rho}\right|$ of primitive integral vectors in the edge $E_{\rho}$. The points of intersection are called points at infinity. Let $\mathcal{C} \in\left|D_{N}\right|$ denote the compactification of the open spectral curve $\mathcal{C}^{\circ}$, i.e., $\mathcal{C}$ is the closure of $\mathcal{C}^{\circ}$ in $X_{N} . \mathcal{C}$ is called the spectral curve.

### 2.7 Casimirs

Let $\alpha$ be a zig-zag path $\alpha=\mathrm{b}_{1} \rightarrow \mathrm{w}_{1} \rightarrow \mathrm{~b}_{2} \rightarrow \cdots \rightarrow \mathrm{w}_{d} \rightarrow \mathrm{~b}_{1}$ in $Z_{\rho}$. We define the Casimir $C_{\alpha}$ by

$$
C_{\alpha}:=(-1)^{d}[\epsilon]([\alpha])[w t]([\alpha])
$$

The Casimirs determine points at infinity of $\mathcal{C}$ as follows: since $[\alpha]$ is primitive and $\left\langle u_{\rho},[\alpha]\right\rangle=0$, we can extend it to a basis $\left(x_{1}, x_{2}\right)$ of M with $[\alpha]=x_{1}$ and $\left\langle x_{2}, u_{\rho}\right\rangle=1$. The affine open variety in $X_{N}$ corresponding to the cone $\rho$ is

$$
U_{\rho}=\operatorname{Spec} \mathbb{C}\left[x_{1}^{ \pm 1}, x_{2}\right] \cong \mathbb{C}^{\times} \times \mathbb{C}
$$

and $D_{\rho} \cap U_{\rho}$ is defined by $x_{2}=0$, and so the character $x_{1}^{-1}=\chi^{-[\alpha]}$ is a coordinate on the dense open torus $\mathbb{C}^{\times}=D_{\rho} \cap U_{\rho}$ in $D_{\rho}$. Therefore, the equation

$$
\begin{equation*}
\chi^{-[\alpha]}\left(\nu_{\rho}(\alpha)\right)=C_{\alpha} \tag{8}
\end{equation*}
$$

defines a point $\nu_{\rho}(\alpha)$ in $D_{\rho}$. In other words, the point is defined as the unique point on the line at infinity such that the monomial $z^{i} w^{j}$, where $-[\alpha]=(i, j)$, evaluates to $C_{\alpha}$. We will prove later (see (41)) that these are precisely the points at infinity of $\mathcal{C}$.

Example 2.4. Consider the fundamental domain of the square lattice, whose zig-zag paths were listed in Example 2.1 and Figure 3. The Casimirs are

$$
\begin{equation*}
C_{\alpha}=-\frac{B}{A X_{1}}, \quad C_{\beta}=-\frac{1}{A B X_{2}}, \quad C_{\gamma}=-\frac{A X_{1} X_{2} X_{3}}{B}, \quad C_{\delta}=-\frac{A B}{X_{3}} \tag{9}
\end{equation*}
$$

Let us denote the normal ray in $\Sigma$ of a zig-zag path $\omega$ by $\rho(\omega)$, so $u_{\rho(\alpha)}=(-1,-1)$ etc. We choose $x_{2}=\chi^{(0,-1)}$ so that $\left\langle(0,-1), u_{\rho(\alpha)}\right\rangle=1$. Then we have $U_{\rho(\alpha)}=\operatorname{Spec} \mathbb{C}\left[x_{1}=z^{-1} w, x_{2}=w^{-1}\right]$ and $D_{\rho(\alpha)} \subset U_{\rho(\alpha)}$ is given by $x_{2}=0$. In this case, $D_{N}=D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}$ and $P(z, w)$ is a global section of $\mathcal{O}_{X_{N}}\left(D_{N}\right)$. We trivialize $\mathcal{O}_{X_{N}}\left(D_{N}\right)$ over $U_{\rho(\alpha)}$ as follows:

$$
\begin{aligned}
\left.\mathcal{O}_{X_{N}}\left(D_{N}\right)\right|_{U_{\rho(\alpha)}}=\left\{t \in \mathbb{C}\left[z^{ \pm 1}, w^{ \pm 1}\right]:\left.\operatorname{div} t\right|_{U_{\rho(\alpha)}}+D_{\rho(\alpha)} \geq 0\right\} & \cong \mathcal{O}_{U_{\rho(\alpha)}} \\
t & \mapsto t x_{2}
\end{aligned}
$$

Then making the change of variables $z=\frac{1}{x_{1} x_{2}}$ and $w=\frac{1}{x_{2}}$, and multiplying by $x_{2}$, the portion of the spectral curve $\mathcal{C}$ in $U_{\rho}$ is cut out by

$$
\left(1+X_{1}+\frac{1}{X_{2}}+X_{1} X_{3}\right) x_{2}-B-\frac{X_{1} X_{3}}{B} x_{2}^{2}-\frac{x_{1} x_{2}^{2}}{A X_{2}}-\frac{A X_{1}}{x_{1}}
$$

so that $\mathcal{C} \cap D_{\rho(\alpha)}$ is given by

$$
-B-\frac{A X_{1}}{x_{1}}=0
$$

Therefore, $\nu(\alpha)$ is given by $\frac{z}{w}=\frac{1}{x_{1}}=C_{\alpha}$, which agrees with (8). The table below lists the points at infinity for each of the zig-zag paths.

| Zig-zag path | Homology class | Basis $x_{1}, x_{2}$ | Point at infinity |
| :---: | :---: | :---: | :---: |
| $\alpha$ | $(-1,1)$ | $(-1,1),(0,-1)$ | $x_{1}=\frac{1}{C_{\alpha}}, x_{2}=0$ |
| $\beta$ | $(-1,-1)$ | $(-1,-1),(0,-1)$ | $x_{1}=\frac{1}{C_{\beta}}, x_{2}=0$ |
| $\gamma$ | $(1,-1)$ | $(1,-1),(0,1)$ | $x_{1}=\frac{1}{C_{\gamma}}, x_{2}=0$ |
| $\delta$ | $(1,1)$ | $(1,1),(0,1)$ | $x_{1}=\frac{1}{C_{\delta}}, x_{2}=0$ |

### 2.8 The spectral transform

Our next goal is to define the spectral transform, which plays the key role in this paper. We present two equivalent definitions of the spectral transform. The first is the original definition of Kenyon and Okounkov KO06, and it is the one which we use in computations. However, it depends on the choice of the distinguished white vertex $\mathbf{w}$. The second is more invariant, and does not require choosing a distinguished white vertex $\mathbf{w}$.

Recall that for each edge $E_{\rho}$ of $N$, we have $\# Z_{\rho}=\# \mathcal{C} \cap D_{\rho}$, but there is no canonical bijection between these sets. We define a parameterization of the points at infinity by zig-zag paths to be a choice of bijections $\nu=\left\{\nu_{\rho}\right\}_{\rho \in \Sigma(1)}$, where

$$
\begin{equation*}
\nu_{\rho}: Z_{\rho} \xrightarrow{\sim} \mathcal{C} \cap D_{\rho} \tag{11}
\end{equation*}
$$

For a curve $\mathcal{C} \in\left|D_{N}\right|$, we denote by $\operatorname{Div}_{\infty}(\mathcal{C})$ the abelian group of divisors on $\mathcal{C}$ supported at the infinity, that is at $\mathcal{C} \cap D_{N}$.

Compactifications of the Kasteleyn operator will play a important role in this paper. The main ingredient in the construction of these compactifications is a combinatorial object called the discrete Abel map introduced by Fock [Foc15] that encodes intersections with zig-zag paths. Let $\Gamma$ be a minimal bipartite graph in $\mathbb{T}$ with Newton polygon $N$ and spectral curve $\mathcal{C}$. The discrete Abel map

$$
\mathbf{d}: B \cup W \cup F \rightarrow \operatorname{Div}_{\infty}(\mathcal{C})
$$

assigns to each vertex and face of $\Gamma$ a divisor at infinity. It is defined uniquely up to a constant by the requirement that for a path $\gamma$ from $x$ to $y$, contained in the fundamental domain $R$, where $x$ and $y$ are either vertices or faces of $\Gamma$, we have

$$
\mathbf{d}(y)-\mathbf{d}(x)=\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}}(\alpha, \gamma)_{R} \nu_{\rho}(\alpha)
$$

Here $(\alpha, \gamma)_{R}$ is the intersection index in $R$, i.e., the signed number of intersections of $\alpha$ with $\gamma$. Since we require $\gamma$ to be contained in $R$, this is well-defined, independent of the choice of path $\gamma$. Locally, the rule is as follows:

1. If b is a black vertex incident to a face $f$, and b and $f$ are separated by $\alpha \in Z_{\rho}$, then $\mathbf{d}(\mathrm{b})=\mathbf{d}(f)+\nu_{\rho}(\alpha)$.
2. If w is a white vertex incident to a face $f$, and w and $f$ are separated by $\alpha \in Z_{\rho}$, then $\mathbf{d}(\mathrm{w})=\mathbf{d}(f)-\nu_{\rho}(\alpha)$.

We normalize $\mathbf{d}$, setting the value of $\mathbf{d}$ at certain face $f_{0}$ of $\Gamma$ to be 0 . Then for any black vertex b, face $f$, and white vertex w of $\widetilde{\Gamma}$ we have:

$$
\begin{equation*}
\operatorname{deg} \mathbf{d}(b)=1, \quad \operatorname{deg} \mathbf{d}(f)=0, \quad \operatorname{deg} \mathbf{d}(w)=-1 \tag{12}
\end{equation*}
$$

Remark 2.5. Only differences of the form $\mathbf{d}(y)-\mathbf{d}(x)$ will appear in our constructions later, so the choice of normalization does not play a role.

Example 2.6. Let us compute the discrete Abel map d for the square lattice in Figure 5. We normalize $\mathbf{d}\left(f_{1}\right)=0$. Then we have

$$
\mathbf{d}\left(\mathrm{b}_{1}\right)=\nu_{\rho(\gamma)}(\gamma), \quad \mathbf{d}\left(\mathrm{b}_{2}\right)=\nu_{\rho(\alpha)}(\alpha), \quad \mathbf{d}\left(\mathrm{w}_{1}\right)=-\nu_{\rho(\beta)}(\beta), \quad \mathbf{d}\left(\mathrm{w}_{2}\right)=-\nu_{\rho(\delta)}(\delta),
$$

where $\nu$ is shown in table (10).
Definition 1. A divisor spectral data related to a Newton polygon $N$ is a triple $(\mathcal{C}, S, \nu)$ where $\mathcal{C} \in\left|D_{N}\right|$ is a genus $g$ curve on the toric surface $X_{N}, S$ is a degree $g$ effective divisor in $\mathcal{C}^{\circ}$, and $\nu=\left\{\nu_{\rho}\right\}$ are parameterizations of the divisors $D_{\rho} \cap \mathcal{C}$, see (11). Denote by $\mathcal{S}_{N}$ the moduli space parameterizing the divisor spectral data on $N$. Let us fix a distinguished white vertex $\mathbf{w}$ of $\Gamma$. Then there is a rational map (here and in the sequel, $\rightarrow$ means a rational map), called the spectral transform, defined by Kenyon and Okounkov [KO06],

$$
\begin{equation*}
\kappa_{\Gamma, \mathbf{w}}: \mathcal{X}_{N} \longrightarrow \mathcal{S}_{N} \tag{13}
\end{equation*}
$$

defined on the dense open subset $H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$of $\mathcal{X}_{N}$ by $[w t] \mapsto(\mathcal{C}, S, \nu)$ as follows:

1. $\mathcal{C}$ is the spectral curve.
2. For generic $[w t], \mathcal{C}$ is a smooth curve and coker $K$ is the pushforward of a line bundle on $\mathcal{C}^{\circ}$. Let $s_{\mathbf{w}}$ be the section of coker $K$ given by the w-entry of the cokernel map. $S$ is defined to be the divisor of this section. In Corollary 6.3, we show that $S$ has degree $g$. Then $S$ is the set of $g$ points in $\mathcal{C}^{\circ}$ where the $\mathbf{w}$-column of the adjugate matrix $Q=Q(z, w)=K^{-1} \operatorname{det} K$ vanishes.
3. The parameterization of points at infinity by zig-zag paths $\nu$ is defined as follows: $\nu_{\rho}(\alpha)$ is the point in $\mathcal{C} \cap D_{\rho}$ satisfying $\chi^{-[\alpha]}=C_{\alpha}$ (see Section 2.7). We call $\nu_{\rho}(\alpha)$ the point at infinity associated to $\alpha$.

Definition 2. A line bundle spectral data related to a Newton polygon $N$ is a triple $(\mathcal{C}, \mathcal{L}, \nu)$ where $\mathcal{C} \in\left|D_{N}\right|$ is a genus $g$ curve on the toric surface $X_{N}, \mathcal{L}$ is a degree $g-1$ line bundle on $\mathcal{C}$, and $\nu$ is a parameterization of points at infinity by zig-zag paths. Denote by $\mathcal{S}_{N}^{\prime}$ the moduli space parameterizing the line bundle spectral data on $N$.

The spectral transform is a rational map

$$
\kappa_{\Gamma, \mathbf{d}}: \mathcal{X}_{N} \longrightarrow \mathcal{S}_{N}^{\prime}
$$

defined on the dense open subset $H^{1}\left(\Gamma, \mathbb{C}^{\times}\right)$of $\mathcal{X}_{N}$ by $[w t] \mapsto(\mathcal{C}, \mathcal{L}, \nu)$, where:

1. $\mathcal{C}$ is the spectral curve.
2. Let $\left.K\right|_{\mathcal{C}^{\circ}}$ denote the restriction of the Kasteleyn matrix to $\mathcal{C}^{\circ}$. The discrete Abel map d determines an extension $\bar{K}$ of $\left.K\right|_{\mathcal{C}^{\circ}}$ to a morphism of locally free sheaves on $\mathcal{C}$; see Section 6 . The coherent sheaf $\mathcal{L}$ is defined as the cokernel of $\bar{K}$. When $\mathcal{C}$ is a smooth curve, which happens for generic $[w t], \mathcal{L}$ is a line bundle. The convention $\operatorname{deg} \mathbf{d}(\mathrm{w})=-1$ implies that $\operatorname{deg} \mathcal{L}=g-1$; see Proposition 6.4.
3. The parameterizations of the divisors $D_{N} \cap \mathcal{C}$ are defined by associating to a zig-zag path $\alpha$ the point at infinity $\nu_{\rho}(\alpha)$.
Since $\rho$ is determined by $\alpha$, we will use the simpler notation $\nu(\alpha):=\nu_{\rho}(\alpha)$ hereafter.
The two types of spectral data are equivalent. Given a degree $g$ effective divisor $S$, we have (Proposition 6.4)

$$
\begin{equation*}
\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(S+\mathbf{d}(\mathbf{w})) \tag{14}
\end{equation*}
$$

On the other hand, given a line bundle $\mathcal{L}$ and a white vertex $\mathbf{w}$, we can recover $S$ as follows. Consider the Abel-Jacobi map

$$
\begin{aligned}
A^{g}: \operatorname{Sym}^{g} \mathcal{C} & \rightarrow \operatorname{Jac}(\mathcal{C}) \\
E & \mapsto \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(E+\mathbf{d}(\mathbf{w}))
\end{aligned}
$$

Then $A^{g}$ is birational by the Abel-Jacobi theorem [Bea13, Corollary 4.6]. We obtain $S=\left(A^{g}\right)^{-1}\left(\mathcal{O}_{\mathcal{C}}\right)$.
Example 2.7. We compute the spectral transform for our running example of the square lattice. Let us take the distinguished white vertex to be $\mathbf{w}=\mathrm{w}_{1}$.

$$
Q=\left(\begin{array}{cc}
\mathrm{w}_{1} & \mathrm{w}_{2}  \tag{15}\\
X_{1}-\frac{1}{A X_{2} z} & -1+\frac{X_{1} X_{3}}{B w} \\
1-B w & 1-A z
\end{array}\right) \begin{aligned}
& \mathrm{b}_{1} \\
& \mathrm{~b}_{2}
\end{aligned} .
$$

Solving $Q_{\mathrm{b}_{1} \mathbf{w}}(p, q)=Q_{\mathrm{b}_{2} \mathbf{w}}(p, q)=0$, we get

$$
\begin{equation*}
p=\frac{1}{A X_{1} X_{2}}, \quad q=\frac{1}{B} \tag{16}
\end{equation*}
$$

Therefore, the spectral transform is:

$$
\begin{aligned}
\kappa_{\Gamma, \mathbf{w}}: H^{1}\left(\Gamma, \mathbb{C}^{\times}\right) & -\mathcal{S}_{N} \\
\left(X_{1}, X_{2}, X_{3}, A, B\right) & \mapsto(\mathcal{C},(p, q), \nu)
\end{aligned}
$$

where $\mathcal{C}=\{P(z, w)=0\}$ with $P(z, w)$ as is in (5), $S=(p, q)$ is a single point (the genus $g=1$ since $N$ in Figure 3 has one interior lattice point) and $\nu$ is as shown in table (10).

## 3 The main theorem

Below we introduce functions $\mathrm{V}_{\mathrm{bw}}$ on the moduli space $\mathcal{S}_{N}$ of spectral data, relying on results in the remaining Sections 5, 6, 7. They are defined for any pair (b, w) $\in B \times W$ of black and white vertices, and defined as the solution to a system of linear equations $\mathbb{V}_{\text {bw }}$.

The main result of the paper is the following.
Theorem 3.1. For the distinguished white vertex $\mathbf{w}$, the pull-back of the function $\mathrm{V}_{\mathrm{bw}}$ under the spectral map coincides, up to a multiplicative constant, with the bw matrix element $Q_{\mathrm{bw}}$ of the adjugate matrix $Q:=K^{-1} \operatorname{det} K$ of the Kasteleyn matrix $K$. That is,

$$
\begin{equation*}
Q_{\mathrm{bw}}=c \cdot \kappa_{\Gamma, \mathbf{w}}^{*}\left(\mathrm{~V}_{\mathrm{bw}}\right) \tag{17}
\end{equation*}
$$

where $c$ depends on b (and $\mathbf{w}$ ).
As an application of this result, we get an explicit description of the inverse to the spectral map (13); see Section 3.2

The next few sections discuss the structure of the system of linear equations $\mathbb{V}_{\text {bw }}$. Detailed examples are given in Section 4.

### 3.1 The matrix $\mathbb{V}_{b w}$

The system of linear equations $\mathbb{V}_{\mathrm{bw}}$ is in the variables $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$ where $N_{\mathrm{bw}} \subset \mathrm{M}_{\mathbb{R}}$ is a convex polygon, introduced in Section 3.1.1.2, and called the small Newton polygon. There is one system for every pair $(\mathrm{b}, \mathrm{w}) \in B \times W$. The system $\mathbb{V}_{\mathrm{bw}}$ is of the form (matrix) $\left(a_{m}\right)=0$; we also denote this matrix by $\mathbb{V}_{\text {bw }}$. Therefore, the columns of the matrix $\mathbb{V}_{\text {bw }}$ are indexed by the lattice points $N_{\mathrm{bw}} \cap \mathrm{M}$. By Corollary 5.3, the polygon $N_{\mathrm{bw}}$ is the Newton polygon of the Laurent polynomial $Q_{\mathrm{bw}}$.

The equations in $\mathbb{V}_{\text {bw }}$, i.e., the rows of the matrix $\mathbb{V}_{\text {bw }}$ are defined in Section 3.1.2, There are two types:

1. There is a row for each of the points $\left(p_{1}, q_{1}\right), \ldots,\left(p_{g}, q_{g}\right)$ of the divisor $S$ on the spectral curve. The entry of the row in column $m \in N_{\mathrm{bw}} \cap \mathrm{M}$ is $\chi^{m}\left(p_{i}, q_{i}\right)$.
2. The remaining rows correspond to certain zig-zag paths $\alpha$. The entries in the row corresponding to $\alpha$ are certain monomials in $C_{\alpha}$.
Let us proceed to the precise definition of the matrix $\mathbb{V}_{\text {bw }}$.

### 3.1.1 Columns of the matrix $\mathbb{V}_{b w}$

We now describe the small Newton polygons, whose lattice points correspond to columns of $\mathbb{V}_{\text {bw }}$.
3.1.1.1 Rational Abel map D. Recall the set $\left\{D_{\rho}\right\}$ of lines at infinity of the toric surface $X_{N}$. Consider the $\mathbb{Q}$-vector space $\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}\left(X_{N}\right)$ of $\mathbb{Q}$-divisors at infinity, defined as the $\mathbb{Q}$-vector space with a basis given by the divisors $D_{\rho}$ :

$$
\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}\left(X_{N}\right):=\bigoplus_{\rho \in \Sigma(1)} \mathbb{Q} D_{\rho} .
$$

We define a rational Abel map

$$
\mathbf{D}: V \rightarrow \operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}\left(X_{N}\right)
$$

which assigns to each vertex v of the graph $\Gamma$ a $\mathbb{Q}$-divisor at infinity $\mathbf{D}(\mathrm{v})$ as follows:

1. Normalize $\mathbf{D}(\mathbf{w})=0$. As in the case of $\mathbf{d}$, the choice of normalization plays no role, and we can replace 0 with any $\mathbb{Q}$-divisor.
2. For any path $\gamma$ contained in $R$ from $\mathrm{v}_{1}$ to $\mathrm{v}_{2}$,

$$
\mathbf{D}\left(\mathrm{v}_{2}\right)-\mathbf{D}\left(\mathrm{v}_{1}\right)=\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}} \frac{(\alpha, \gamma)_{R}}{\left|E_{\rho}\right|} D_{\rho}
$$

where $(\cdot, \cdot)_{R}$ is the intersection index in $R$, i.e., the signed number of intersections of $\alpha$ with $\gamma$. The following lemma follows from definitions.

Lemma 3.2. Let $\alpha, \beta$ be the zig-zag paths through $e=\mathrm{bw}$, with $\alpha \in Z_{\sigma}, \beta \in Z_{\rho}$. Then, we have

$$
\begin{equation*}
\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})=-\frac{1}{\left|E_{\sigma}\right|} D_{\sigma}-\frac{1}{\left|E_{\rho}\right|} D_{\rho}-\operatorname{div} \phi(e) \tag{18}
\end{equation*}
$$

where $\phi(e)$ is the character defined in (1) and $\operatorname{div} \phi(e)$ denotes its (Weil) divisor as in 48).
3.1.1.2 Small Newton polygons. Recall the divisor $D_{N}$ at infinity of $X_{N}$, see (6). Given an edge $e=\mathrm{bw}$, we define a $\mathbb{Q}$-divisor at infinity

$$
\begin{equation*}
Y_{\mathrm{bw}}:=D_{N}-\mathbf{D}(\mathrm{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho} \tag{19}
\end{equation*}
$$

Here the double sum is over all zig-zag paths $\alpha$ passing through b and $\left|E_{\rho}\right|$ denotes the number of integral primitive vectors in $E_{\rho}$ as in Section 2.6. We define $b_{\rho} \in \mathbb{Q}$ as the multiplicities of the projective lines at infinity $D_{\rho}$ in the divisor $Y_{\mathrm{bw}}$ :

$$
\begin{equation*}
Y_{\mathrm{bw}}=\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho} \tag{20}
\end{equation*}
$$



Figure 6: The two small polygons in Example 3.4. The big black dot denotes the origin, while the other black dots are integral points.

Definition 3.3. The small Newton polygon $N_{\text {bw }}$ is the polygon defined by the formula

$$
\begin{equation*}
N_{\mathrm{bw}}=\bigcap_{\rho \in \Sigma(1)}\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle \geq-b_{\rho}\right\} \tag{21}
\end{equation*}
$$

There is a canonical bijection between divisors $D$ in $\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}\left(X_{N}\right)$ and convex polygons $P$ with rational intercepts (see Proposition A. 3 for its importance in toric geometry):

$$
\begin{equation*}
D=\sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho} \leftrightarrow P=\bigcap_{\rho \in \Sigma(1)}\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle \geq-a_{\rho}\right\}, \quad a_{\rho} \in \mathbb{Q} \tag{22}
\end{equation*}
$$

Therefore, $N_{\mathrm{bw}}$ is the polygon associated to the divisor $Y_{\mathrm{bw}}$ in (22).
The polygon $N_{\text {bw }}$ may not be integral. We will consider only integral points in it. The convex hull of the integral points in $N_{\mathrm{bw}}$ contains the Newton polygon of $Q_{\mathrm{bw}}$ (Corollary 5.3).

Example 3.4. We compute the small polygons for the square lattice in Figure 5. Recall that we chose $\mathbf{w}=\mathrm{w}_{1}$. Since there is only one zig-zag path in each homology direction, the rational Abel $\operatorname{map} \mathbf{D}$ is obtained from $\mathbf{d}$ by replacing the point at infinity with the corresponding line at infinity, so from Example 2.6, we have

$$
\mathbf{D}\left(\mathrm{b}_{1}\right)=D_{\rho(\gamma)}, \quad \mathbf{D}\left(\mathrm{b}_{2}\right)=D_{\rho(\alpha)}, \quad \mathbf{D}\left(\mathrm{w}_{1}\right)=-D_{\rho(\beta)}, \quad \mathbf{D}\left(\mathrm{w}_{2}\right)=-D_{\rho(\delta)}
$$

We have $D_{N}=D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}$, using which we compute

$$
\begin{aligned}
Y_{\mathrm{b}_{1} \mathrm{w}_{1}} & =\left(D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}\right)-\left(-D_{\rho(\beta)}\right)+D_{\rho(\gamma)}-\left(D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}\right) \\
& =D_{\rho(\beta)}+D_{\rho(\gamma)}, \\
Y_{\mathrm{b}_{2} \mathrm{w}_{1}} & =\left(D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}\right)-\left(-D_{\rho(\beta)}\right)+D_{\rho(\alpha)}-\left(D_{\rho(\alpha)}+D_{\rho(\beta)}+D_{\rho(\gamma)}+D_{\rho(\delta)}\right) \\
& =D_{\rho(\alpha)}+D_{\rho(\beta)} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& N_{\mathrm{b}_{1} \mathrm{w}_{1}}=\{-i-j \geq 0\} \cap\{i-j \geq-1\} \cap\{i+j \geq-1\} \cap\{-i+j \geq 0\}, \\
& N_{\mathrm{b}_{2} \mathrm{w}_{1}}=\{-i-j \geq-1\} \cap\{i-j \geq-1\} \cap\{i+j \geq 0\} \cap\{-i+j \geq 0\},
\end{aligned}
$$

see Figure 6. Note that the convex hulls of the lattice points are the Newton polygons of $Q_{\mathrm{b}_{1} \mathrm{w}_{1}}$ and $Q_{\mathrm{b}_{2} \mathrm{w}_{1}}$ in (15).

### 3.1.2 Rows of the matrix $\mathbb{V}_{b w}$

Recall that the variables in $\mathbb{V}_{\mathrm{bw}}$ are $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$. We identify a Laurent polynomial $F=$ $\sum_{m \in \mathrm{M}} b_{m} \chi^{m}$ with its vector of coefficients $\left(b_{m}\right)_{m \in \mathrm{M}}$. The equations in $\mathbb{V}_{\mathrm{bw}}$ are of two types:

1. For each $1 \leq i \leq g$, we have the linear equations

$$
\begin{equation*}
\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m}\left(p_{i}, q_{i}\right)=0 \tag{23}
\end{equation*}
$$

so the entry of the corresponding row of $\mathbb{V}_{\text {bw }}$ in column $m$ is $\chi^{m}\left(p_{i}, q_{i}\right)$.
2. Recall the notation $\lfloor x\rfloor$ for the largest integer $n$ such that $n \leq x$.

Given a $\mathbb{Q}$-divisor $D=\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$, we define a divisor with integral coefficients

$$
\lfloor D\rfloor:=\sum_{\rho \in \Sigma(1)}\left\lfloor b_{\rho}\right\rfloor D_{\rho} .
$$

Recall the divisor $Y_{\mathrm{bw}}$ in (19). For a divisor $D$ at infinity, let $\left.D\right|_{\mathcal{C}}$ denote the divisor corresponding to the intersection of $D$ with $\mathcal{C}$. Precisely, if $D=\sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}$, then $\left.D\right|_{\mathcal{C}}:=$ $\sum_{\rho \in \Sigma(1)} a_{\rho} \sum_{\alpha \in Z_{\rho}} \nu(\alpha)$. We have a linear equation for every zig-zag path $\alpha$ such that $\nu(\alpha)$ appears in

$$
\begin{equation*}
-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)+\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}} \tag{24}
\end{equation*}
$$

Suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes an equation. We extend $[\alpha]$ to a basis $\left(x_{1}, x_{2}\right)$ of M , where $x_{1}:=[\alpha]$ and $\left\langle x_{2}, u_{\rho}\right\rangle=1$, so that for any $m \in \mathrm{M}$, we can write

$$
\chi^{m}=x_{1}^{b_{m}} x_{2}^{c_{m}}, \quad b_{m}, c_{m} \in \mathbb{Z}
$$

Let $N_{\mathrm{bw}}^{\rho}$ be the set of lattice points in $N_{\mathrm{bw}}$ closest to the edge $E_{\rho}$ of $N$ i.e., the set of points in $N_{\mathrm{bw}}$ that minimize the functional $\left\langle *, u_{\rho}\right\rangle$. Then the equation associated with $\alpha$ is

$$
\begin{equation*}
\sum_{m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}} a_{m} C_{\alpha}^{-b_{m}}=0 \tag{25}
\end{equation*}
$$

So the entry in column $m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}$ is the monomial $C_{\alpha}^{-b_{m}}$, and the entries in the other columns are 0 . Choosing a different basis vector $x_{2}$ leads to the same equation multiplied by a monomial in $C_{\alpha}$.
Remark 3.5. When all the sides of the Newton polygon are primitive, we call the Newton polygon simple. In this case, we have $\left[Y_{\mathrm{bw}}\right]=Y_{\mathrm{b} \mathbf{w}}$ and $\mathbf{d}(\mathbf{w})-\mathbf{d}(\mathrm{b})=\left.(\mathbf{D}(\mathbf{w})-\mathbf{D}(\mathrm{b}))\right|_{\mathcal{C}}$. Then Formula (24) simplifies considerably to

$$
\begin{equation*}
\sum_{\alpha \in Z: \mathrm{b} \notin \alpha} \nu(\alpha) . \tag{26}
\end{equation*}
$$

So for a simple Newton polygon the Casimir rows of the matrix $\mathbb{V}_{b w}$, i.e., the rows providing equations (25), are parameterized by the zig-zag paths $\alpha$ which do not contain the vertex b.

### 3.1.3 The functions $\mathrm{V}_{\mathrm{bw}}$

The number of rows of $\mathbb{V}_{b w}$ is at least as large as the number of columns minus one, but not necessarily equal. However, Proposition 7.3 shows that there is a unique solution to $\mathbb{V}_{\text {bw }}$ up to a multiplicative constant. Therefore,

$$
\begin{equation*}
\mathrm{V}_{\mathrm{bw}}:=\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m} \tag{27}
\end{equation*}
$$

is uniquely defined up to a multiplicative constant (where $\left(a_{m}\right)_{m \in N_{\text {bw }} \cap M}$ is a solution to $\mathbb{V}_{\text {bw }}$ ). Only ratios of the values of these functions that are independent of the multiplicative constant appear in the inverse map, see Section 3.2.

Remark 3.6. When the equations in $\mathbb{V}_{\text {bw }}$ are linearly independent (so there is exactly one less equation than the number of variables), we can prepend to $\mathbb{V}_{\mathrm{bw}}$ the equation $\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m}$ to get a square matrix, which we denote by $\mathbb{V}_{\mathrm{bw}}^{\chi}$. Then the function $\mathrm{V}_{\mathrm{bw}}$ is the determinant:

$$
\mathrm{V}_{\mathrm{bw}}=\operatorname{det} \mathbb{V}_{\mathrm{bw}}^{\chi}
$$

Indeed, given an $(n-1) \times n$ matrix $\left(a_{i j}\right)$, the system of linear equations $\sum_{j=1}^{n} a_{i j} x_{j}=0$ has a solution given by the signed maximal minors $A_{j}$ of the matrix $A$ :

$$
x_{j}=(-1)^{j} A_{j}
$$

Here $A_{j}$ is the determinant of the matrix obtained by deleting the $j$-th column of $A$. Therefore, the determinant of the augmented matrix $\mathbb{V}_{\text {bw }}^{\chi}$ recovers the expression $\mathrm{V}_{\mathrm{bw}}$ in (27).

Example 3.7. We compute the linear system of equations $\mathbb{V}_{b w}$ for the square lattice in Figure 5 with $\mathbf{w}=\mathrm{w}_{1}$. Since both black vertices are contained in every zig-zag path, the formula (26) is 0 , so there are no equations of type 2 in $\mathbb{V}_{b \mathbf{w}}$ for $\mathrm{b} \in B$. Therefore,

$$
\mathbb{V}_{\mathrm{b}_{1} \mathbf{w}}=\left(\begin{array}{ll}
1 & p^{-1}
\end{array}\right), \quad \mathbb{V}_{\mathrm{b}_{2} \mathbf{w}}=\left(\begin{array}{ll}
1 & q
\end{array}\right)
$$

By Remark 3.6. we get

$$
\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}=\left|\begin{array}{cc}
1 & z^{-1}  \tag{28}\\
1 & p^{-1}
\end{array}\right|, \quad \mathrm{V}_{\mathrm{b}_{2} \mathbf{w}}=\left|\begin{array}{cc}
1 & w \\
1 & q
\end{array}\right|
$$

Using (16), we have

$$
\begin{aligned}
& \kappa_{\Gamma, \mathbf{w}}^{*}\left(\mathrm{~V}_{\mathrm{b}_{1} \mathbf{w}}\right)=A X_{1} X_{2}-\frac{1}{z}=A X_{2} Q_{\mathrm{b}_{1} \mathbf{w}} \\
& \kappa_{\Gamma, \mathbf{w}}^{*}\left(\mathrm{~V}_{\mathrm{b}_{2} \mathbf{w}}\right)=\frac{1}{B}-w=\frac{1}{B} Q_{\mathrm{b}_{2} \mathbf{w}}
\end{aligned}
$$

verifying the conclusion of Theorem 3.1.

### 3.2 Reconstructing weights via functions $\mathrm{V}_{\mathrm{bw}}$.

Take a white vertex w and a zig-zag path $\alpha$ containing w . The pair ( $\mathrm{w}, \alpha$ ) determines a wedge $W:=\mathrm{b} \xrightarrow{e} \mathrm{w} \xrightarrow{e^{\prime}} \mathrm{b}^{\prime}$, where w is a white vertex incident to the vertices $\mathrm{b}, \mathrm{b}^{\prime}$ such that $\mathrm{bwb}^{\prime}$ is a part of $\alpha$. Recall $\phi(e)$ from (11), and the Kasteleyn $\operatorname{sign} \epsilon(e)$. We assign to this wedge the ratio

$$
\begin{equation*}
r_{W}:=-\frac{\epsilon\left(e^{\prime}\right) \phi\left(e^{\prime}\right) \mathrm{V}_{\mathrm{b}^{\prime} \mathbf{w}}}{\epsilon(e) \phi(e) \mathrm{V}_{\mathrm{b} \mathbf{w}}}(\nu(\alpha)) \tag{29}
\end{equation*}
$$

Note that we use the distinguished white vertex $\mathbf{w}$ in the expression rather than w . The expression is in fact independent of w , as we will see in the proof of Theorem 3.10 below.

Remark 3.8. The ratio on the right is a rational function on the curve. We evaluate the ratio at the point at infinity of the spectral curve $\nu(\alpha)$ corresponding to the zig-zag path $\alpha$, see (11). To do this, we first extend $[\alpha]$ to a basis $\left(x_{1}, x_{2}\right)$ of M with $[\alpha]=x_{1}$ and $\left\langle x_{2}, u_{\rho}\right\rangle=1$, as explained in Section 2.7 Then $\nu(\alpha)$ is given by $\frac{1}{x_{1}}=C_{\alpha}, x_{2}=0$. The numerator and denominator in (29) vanish to the same order in $x_{2}$ by Corollary 6.2 below, so after factoring out and canceling the highest power of $x_{2}$ in the numerator and denominator, we can evaluate at $x_{1}=\frac{1}{C_{\alpha}}, x_{2}=0$ to get a well-defined number.

Let $L=\mathrm{b}_{1} \rightarrow \mathrm{w}_{1} \rightarrow \mathrm{~b}_{2} \rightarrow \cdots \rightarrow \mathrm{~b}_{\ell}=\mathrm{b}_{1}$ be an oriented loop on $\Gamma$. It is a concatenation of wedges $W_{i}:=\mathrm{b}_{i-1} \mathrm{w}_{i} \mathrm{~b}_{i}, i=1, \ldots, \ell$ (with $i$ taking values cyclic modulo $\ell$ ) provided by the white vertices. Denote by $\alpha_{i}$ the zig-zag path assigned to the wedge $W_{i}$. We define a cohomology class [ $\omega$ ] by

$$
\begin{equation*}
[\omega]([L]):=\prod_{i=1}^{\ell} r_{W_{i}} \tag{30}
\end{equation*}
$$

Lemma 3.9. The product (30) does not depend on the ambiguities of the multiplicative constants in the involved functions $\mathrm{V}_{\mathrm{bw}}$.

Proof. For each black vertex $\mathrm{b}_{i}$ in $L, \mathrm{~V}_{\mathrm{b}_{i} \mathbf{w}}$ appears twice in (30), once each in the numerator and denominator, and so the multiplicative constants cancel out.

Theorem 3.10. The cohomology classes $[w t]$ and $\kappa_{\Gamma, \mathbf{w}}^{*}[\omega]$ are equal.
Proof. Let $\mathrm{b} \xrightarrow{e} \mathrm{w} \xrightarrow{e^{\prime}} \mathrm{b}^{\prime}$ be a wedge with zig-zag path $\alpha \in Z_{\rho}$. The restriction of the characteristic polynomial $\left.P(z, w)\right|_{D_{\rho}}$ is the partition function of those dimers whose homology class in $N$ lies on $E_{\rho}$. From the explicit construction of external dimers in GK13 (that is, dimers whose homology classes are in $\partial N$ ), we have that each dimer with homology class in $E_{\rho}$ uses exactly one of the edges $e$ or $e^{\prime}$. Since $Q_{\mathrm{bw}}(z, w)$ is the partition function of dimers with the vertices $\mathrm{b}, \mathrm{w}$ removed, we have

$$
\left.P\right|_{D_{\rho}}=\left.w t(e) \epsilon(e) \phi(e) Q_{\mathrm{bw}}\right|_{D_{\rho}}+\left.w t\left(e^{\prime}\right) \epsilon\left(e^{\prime}\right) \phi\left(e^{\prime}\right) Q_{\mathrm{b}^{\prime} \mathrm{w}}\right|_{D_{\rho}}
$$

Since $\nu(\alpha)$ is on the spectral curve, $P(\nu(\alpha))=0$, from which we get

$$
\begin{equation*}
\frac{w t(e)}{w t\left(e^{\prime}\right)}=-\frac{\epsilon\left(e^{\prime}\right) \phi\left(e^{\prime}\right) Q_{\mathrm{b}^{\prime} \mathrm{w}}}{\epsilon(e) \phi(e) Q_{\mathrm{bw}}}(\nu(\alpha)) \tag{31}
\end{equation*}
$$

We have $\operatorname{corank}(K)=1$ at smooth points of $\mathcal{C}$. Note that $\left.K Q\right|_{\mathcal{C}}=0$. Therefore, for generic $w t$, since $\mathcal{C}$ is smooth, $Q$ is a rank 1 matrix given by

$$
Q=\operatorname{ker} K^{*} \otimes \operatorname{coker} K
$$

This implies that

$$
\frac{Q_{\mathrm{bw}}}{Q_{\mathrm{b}^{\prime} \mathrm{w}}}(\nu(\alpha))=\frac{Q_{\mathrm{bw}}}{Q_{\mathrm{b}^{\prime} \mathbf{w}}}(\nu(\alpha))
$$

Example 3.11. Consider the cycle $a$ in Figure 5 given by the red horizontal path. We write it as the concatenation of the two wedges $W_{1}$ and $W_{2}$ represented by $\left(\mathrm{w}_{1}, \delta\right)$ and $\left(\mathrm{w}_{1}, \gamma\right)$ respectively. From

Table (10), we know that in the basis $x_{1}=z w, x_{2}=w$, the point $\nu(\delta)$ is given by $x_{1}=\frac{1}{C_{\delta}}, x_{2}=0$. Using (28), and making the substitution $z=\frac{x_{1}}{x_{2}}, w=x_{2}$, we get

$$
\begin{aligned}
r_{W_{1}} & =-\frac{-1 \cdot w^{-1} \cdot V_{\mathrm{b}_{2} \mathbf{w}}}{-1 \cdot z \cdot V_{\mathrm{b}_{1} \mathbf{w}}}(\nu(\delta)) \\
& =-\frac{1}{z w} \frac{q-w}{p^{-1}-z^{-1}}(\nu(\delta)) \\
& =\frac{-\left(q-x_{2}\right)}{x_{1} p^{-1}-x_{2}}\left(\frac{1}{C_{\delta}}, 0\right) \\
& =-p q C_{\delta}
\end{aligned}
$$

Similarly, from table (10) we know that in the basis $x_{1}=\frac{z}{w}, x_{2}=w$, the point $\nu(\gamma)$ is given by $x_{1}=\frac{1}{C_{\gamma}}, x_{2}=0$. Using (28), and making the substitution $z=x_{1} x_{2}, w=x_{2}$, we get

$$
\begin{aligned}
r_{W_{2}} & =-\frac{1 \cdot 1 \cdot V_{\mathrm{b}_{1} \mathbf{w}}}{-1 \cdot w^{-1} \cdot V_{\mathrm{b}_{2} \mathbf{w}}}(\nu(\gamma)) \\
& =w \frac{p^{-1}-z^{-1}}{q-w}(\nu(\gamma)) \\
& =x_{2} \frac{p^{-1}-\frac{1}{x_{1} x_{2}}}{q-x_{2}}\left(\frac{1}{C_{\gamma}}, 0\right) \\
& =-\frac{C_{\gamma}}{q}
\end{aligned}
$$

Therefore, $[\omega]([a])=p C_{\gamma} C_{\delta}$, and using (9) and (16), we have

$$
\begin{aligned}
\kappa_{\Gamma, \mathbf{w}}^{*}[\omega]([a]) & =\left(\frac{1}{A X_{1} X_{2}}\right) \cdot\left(-\frac{A X_{1} X_{2} X_{3}}{B}\right) \cdot\left(-\frac{A B}{X_{3}}\right) \\
& =A .
\end{aligned}
$$

## 4 Examples

In this section, we work out two detailed examples.

### 4.1 Primitive genus 2 example

Consider the hexagonal graph $\Gamma$ with Newton polygon $N$ and normal fan $\Sigma$ as shown in Figure 7 We label the vertices of $\Gamma$ as in Figure 8. We label the zig-zag paths by $\alpha, \beta, \gamma$, and denote the ray of $\Sigma$ dual to $\tau \in\{\alpha, \beta, \gamma\}$ by $\sigma_{\tau}$.

We can take $X_{i}=[w t]\left(\left[\partial f_{i}\right]\right), i=1, \ldots, 4$, and $A=[w t]([a]), B=[w t]([b])$ as coordinates on $H^{1}(\Gamma, \mathbb{C})$ (see Figure 8).

The Casimirs are

$$
\begin{equation*}
C_{\alpha}=-\frac{B^{2} X_{1} X_{2} X_{4}}{A}, \quad C_{\beta}=-\frac{X_{3}}{A B^{3} X_{1}^{2} X_{4}}, \quad C_{\gamma}=\frac{A^{2} B X_{1}}{X_{2} X_{3}} \tag{32}
\end{equation*}
$$




Figure 7: A hexagonal graph, its Newton polygon $N$ and normal fan $\Sigma$, with zig-zag paths and rays labeled.


Figure 8: Labeling of the vertices and faces of $\Gamma$, a cocycle representing $[w t]$ and $\phi$, where $X_{i}=$ $[w t]\left(\left[\partial f_{i}\right]\right), A=[w t]([a]), B=[w t](b)$, and $a$ and $b$ are the red and green cycles respectively. The Kasteleyn sign $\epsilon$ is 1 for all edges. If no weight or $\phi$ is indicated for an edge, it means that it is 1.


Figure 9: The small polygons for the hexagonal graph.

The Kasteleyn matrix is

$$
K=\left(\begin{array}{ccccc}
\mathrm{b}_{1} & \mathrm{~b}_{2} & \mathrm{~b}_{3} & \mathrm{~b}_{4} & \mathrm{~b}_{5} \\
1 & 0 & 1 & 0 & A z \\
\frac{1}{X_{2}} & X_{3} & 0 & 1 & 0 \\
0 & 1 & \frac{1}{A B X_{1} z w} & 0 & 1 \\
\mathrm{w}_{1} \\
X_{1} X_{4} B w & 0 & 1 & 1 & 0 \\
0 & B w & 0 & 1 & 1
\end{array}\right) \mathrm{w}_{2} \mathrm{w}_{3} \mathrm{w}_{4} \mathrm{w}_{5}
$$

Let $P(z, w)=\operatorname{det} K$ and $\mathcal{C}=\overline{\{P(z, w)=0\}}$. The spectral transform is $\kappa_{\Gamma, \mathbf{w}}=(\mathcal{C}, S, \nu) \in \mathcal{S}_{N}$, where since the interior of $N$ contains two lattice points, the divisor $S=\left(p_{1}, q_{1}\right)+\left(p_{2}, q_{2}\right)$ is a sum
of two points, where

$$
\begin{align*}
& p_{1}=-\frac{\sqrt{\left(-B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}-B\right)^{2}-4 B^{2} X_{1} X_{2} X_{4}}+B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}+B}{2 A B X_{1}}, \\
& q_{1}=\frac{-\sqrt{\left(-B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}-B\right)^{2}-4 B^{2} X_{1} X_{2} X_{4}}+B X_{1} X_{2} X_{3} X_{4}+B X_{1} X_{2} X_{4}+B}{2 B^{2} X_{1} X_{2} X_{4}}, \\
& p_{2}=-\frac{-\sqrt{\left(-B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}-B\right)^{2}-4 B^{2} X_{1} X_{2} X_{4}}+B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}+B}{2 A B X_{1}}, \\
& q_{2}=\frac{\sqrt{\left(-B X_{1} X_{2} X_{3} X_{4}-B X_{1} X_{2} X_{4}-B\right)^{2}-4 B^{2} X_{1} X_{2} X_{4}}+B X_{1} X_{2} X_{3} X_{4}+B X_{1} X_{2} X_{4}}{2 B^{2} X_{1} X_{2} X_{4}} . \tag{33}
\end{align*}
$$

The points at infinity are given by the following table:

| Zig-zag path | Homology class | Basis $x_{1}, x_{2}$ | Point at infinity |
| :---: | :---: | :---: | :---: |
| $\alpha$ | $(-1,2)$ | $(-1,2),(0,-1)$ | $x_{1}=\frac{1}{C_{\alpha}}, x_{2}=0$ |
| $\beta$ | $(-1,-3)$ | $(-1,-3),(0,-1)$ | $x_{1}=\frac{1}{C_{\beta}}, x_{2}=0$ |
| $\gamma$ | $(2,1)$ | $(2,1),(-1,0)$ | $x_{1}=\frac{1}{C_{\gamma}}, x_{2}=0$ |

The discrete Abel map $\mathbf{D}$ is given by
$\mathbf{D}(\mathbf{w})=0$,
$\mathbf{D}\left(\mathrm{b}_{1}\right)=D_{\beta}+D_{\gamma}$,
$\mathbf{D}\left(\mathrm{b}_{2}\right)=-D_{\alpha}+2 D_{\beta}+D_{\gamma}$,
$\mathbf{D}\left(\mathrm{b}_{3}\right)=D_{\alpha}+D_{\beta}$,
$\mathbf{D}\left(\mathrm{b}_{4}\right)=2 D_{\beta}$,
$\mathbf{D}\left(\mathrm{b}_{5}\right)=-D_{\alpha}+3 D_{\beta}$,
and $D_{N}=2 D_{\alpha}+2 D_{\beta}+D_{\gamma}$. Since $\mathbf{D}(\mathbf{w})=0$ and every black vertex b is contained in every zig-zag path, we have

$$
\begin{aligned}
Y_{\mathrm{bw}} & =2 D_{\alpha}+2 D_{\beta}+D_{\gamma}+\mathbf{D}(\mathrm{b})-D_{\alpha}-D_{\beta}-D_{\gamma} \\
& =\mathbf{D}(\mathrm{b})+D_{\alpha}+D_{\beta}
\end{aligned}
$$

Using this, we compute

$$
\begin{array}{lll}
Y_{\mathrm{b}_{1} \mathbf{w}}=D_{\alpha}+2 D_{\beta}+D_{\gamma}, & Y_{\mathrm{b}_{2} \mathbf{w}}=3 D_{\beta}+D_{\gamma}, & Y_{\mathrm{b}_{3} \mathbf{w}}=2 D_{\alpha}+2 D_{\beta} \\
Y_{\mathrm{b}_{4} \mathbf{w}}=D_{\alpha}+3 D_{\beta}, & Y_{\mathrm{b}_{5} \mathbf{w}}=4 D_{\beta}
\end{array}
$$

The small polygons are shown in Figure 9 Since the Newton polygon $N$ is primitive, we are in the setting of Remark 3.5. Every zig-zag path contains every black vertex, so the expression (26) is 0 . Therefore, there are no equations of type 2 in the linear system $\mathbb{V}_{b w}$ for any black vertex $b$. Since $g=2$, we have two equations of type 1 for every black vertex b . Moreover, we note that each of the small polygons in Figure 9 contains exactly three lattice points, so by Remark 3.6, we get

$$
\begin{array}{ll}
\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}=\left|\begin{array}{ccc}
1 & w & z^{-1} w^{-1} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|, & \mathrm{V}_{\mathrm{b}_{2} \mathbf{w}}=\left|\begin{array}{ccc}
1 & z^{-1} & z^{-1} w^{-1} \\
1 & p_{1}^{-1} & p_{1}^{-1} q_{1}^{-1} \\
1 & p_{2}^{-1} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|, \\
\mathrm{V}_{\mathrm{b}_{4} \mathbf{w}}=\left|\begin{array}{ccc}
1 & w & z^{-1} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|, & \mathrm{V}_{\mathrm{b}_{3} \mathbf{w}}=\left|\begin{array}{ccc}
1 & w & w^{2} \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|,
\end{array}
$$

The boundary of the face $f_{2}$ is the concatenation of the three wedges $W_{1}, W_{2}$ and $W_{3}$ represented by $\left(\mathrm{w}_{2}, \alpha\right),(\mathbf{w}, \beta)$ and $\left(\mathrm{w}_{4}, \gamma\right)$ respectively. We compute

$$
r_{W_{1}}=-\frac{V_{\mathrm{b}_{1} \mathrm{w}_{2}}}{V_{\mathrm{b}_{4} \mathrm{w}_{2}}}(\nu(\alpha))=-\frac{\left|\begin{array}{ccc}
1 & w & z^{-1} w^{-1} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & w & z^{-1} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}(\nu(\alpha))
$$

To evaluate at $\nu(\alpha)$, as explained in Remark [3.8, we extend $[\alpha]=(-1,2)$ to the basis $\left(x_{1}, x_{2}\right)$ of M , where $x_{1}=[\alpha]=(-1,2)$ and $x_{2}=(0,-1)$. Then $\nu(\alpha)$ is given by $x_{1}=\frac{1}{C_{\alpha}}, x_{2}=0$. Expressing $z, w$ in the basis $\left(x_{1}, x_{2}\right)$ as $z=\frac{1}{x_{1} x_{2}^{2}}, w=\frac{1}{x_{2}}$, we get

$$
\begin{aligned}
r_{W_{1}} & =-\frac{\left|\begin{array}{ccc}
1 & \frac{1}{x_{2}} & x_{1} x_{2}^{3} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & \frac{1}{x_{2}} & x_{1} x_{2}^{2} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}\left(\frac{1}{C_{\alpha}}, 0\right)=-\frac{\left|\begin{array}{ccc}
x_{2} & 1 & x_{1} x_{2}^{4} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
x_{2} & 1 & x_{1} x_{2}^{3} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}\left(\frac{1}{C_{\alpha}}, 0\right)=-\frac{\left|\begin{array}{ccc}
0 & 1 & 0 \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & 1 & 0 \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|} \\
& =-\frac{p_{1} q_{1}-p_{2} q_{2}}{q_{1} q_{2}\left(p_{1}-p_{2}\right)},
\end{aligned}
$$

where we factored out $x_{2}$ from the numerator and denominator and then evaluated at $\left(x_{1}, x_{2}\right)=$ $\left(\frac{1}{C_{\alpha}}, 0\right)$.

For $W_{2}$, letting $\left(x_{1}, x_{2}\right)=((-1,-3),(0,-1))$ we have $z=\frac{x_{2}^{3}}{x_{1}}, w=\frac{1}{x_{2}}$, and $\nu(\beta)$ is given by $x_{1}=\frac{1}{C_{\beta}}, x_{2}=0$. Therefore, we get

$$
\begin{aligned}
r_{W_{2}} & =-\frac{V_{\mathrm{b}_{3} \mathbf{w}}}{V_{\mathrm{b}_{1} \mathrm{w}}}(\nu(\beta))=-\frac{\left|\begin{array}{ccc}
1 & w & w^{2} \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & w & z^{-1} w^{-1} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}(\nu(\beta))=-\frac{\left|\begin{array}{ccc}
1 & \frac{1}{x_{2}} & \frac{1}{x_{2}^{2}} \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & \frac{1}{x_{2}} & \frac{x_{1}}{x_{2}^{2}} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}\left(\frac{1}{C_{\beta}}, 0\right)=-\frac{\left|\begin{array}{ccc}
0 & 0 & 1 \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & 0 & \frac{1}{C_{\beta}} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|} \\
& =-C_{\beta} .
\end{aligned}
$$

Finally, for $W_{3}$, letting $\left(x_{1}, x_{2}\right)=((2,1),(-1,0))$ we have $z=\frac{1}{x_{2}}, w=x_{1} x_{2}^{2}$, and $\nu(\gamma)$ is given by
$x_{1}=\frac{1}{C_{\gamma}}, x_{2}=0$. Therefore, we get

$$
\begin{aligned}
r_{W_{3}} & =-\frac{V_{\mathrm{b}_{4} \mathbf{w}}}{V_{\mathrm{b}_{3} \mathbf{w}}}(\nu(\gamma))=-\frac{\left|\begin{array}{ccc}
1 & w & z^{-1} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & w & w^{2} \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}(\nu(\gamma))=-\frac{\left|\begin{array}{ccc}
1 & x_{1} x_{2}^{2} & x_{2} \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & x_{1} x_{2}^{2} & x_{1}^{2} x_{2}^{4} \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}\left(\frac{1}{C_{\gamma}}, 0\right)=-\frac{\left|\begin{array}{ccc}
1 & 0 & 0 \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & 0 & 0 \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|} \\
& =\frac{p_{1} q_{1}-p_{2} q_{2}}{p_{1} p_{2} q_{1} q_{2}\left(q_{1}-q_{2}\right)} .
\end{aligned}
$$

Putting everything together, we get

$$
\begin{aligned}
X_{2} & =-\frac{\left|\begin{array}{ccc}
0 & 1 & 0 \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & 1 & 0 \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|} \frac{\left|\begin{array}{ccc}
0 & 0 & 1 \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & 0 & \frac{1}{C_{\beta}} \\
1 & q_{1} & p_{1}^{-1} q_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1} q_{2}^{-1}
\end{array}\right|} \frac{\left|\begin{array}{ccc}
1 & 0 & 0 \\
1 & q_{1} & p_{1}^{-1} \\
1 & q_{2} & p_{2}^{-1}
\end{array}\right|}{\left|\begin{array}{ccc}
1 & 0 & 0 \\
1 & q_{1} & q_{1}^{2} \\
1 & q_{2} & q_{2}^{2}
\end{array}\right|} \\
& =\frac{C_{\beta}\left(p_{1} q_{1}-p_{2} q_{2}\right)^{2}}{p_{1} p_{2} q_{1}^{2} q_{2}^{2}\left(p_{1}-p_{2}\right)\left(q_{1}-q_{2}\right)},
\end{aligned}
$$

with similar formulas for $X_{1}, X_{3}, X_{4}, A, B$. It may be easily verified that these invert the spectral transform by plugging in the formulas (32) and (33) into the right-hand side and simplifying using computer algebra.

### 4.2 Non-primitive example

Consider the square-octagon graph $\Gamma$ with Newton polygon $N$ and normal fan $\Sigma$ as shown in Figure 10. We label the vertices of $\Gamma$ as in Figure 11. We label the rays of $\Sigma$ by $\sigma_{\alpha}, \sigma_{\beta}, \sigma_{\gamma}, \sigma_{\delta}$ and the two zig-zag paths dual to ray $\sigma_{\tau}$ by $\left\{\tau_{1}, \tau_{2}\right\}$, for $\tau \in\{\alpha, \beta, \gamma, \delta\}$.

We can take $X_{i}:=[w t]\left(\left[\partial f_{i}\right]\right), i=1, \ldots, 7$, and $A:=[w t]([a]), B:=[w t]([b])$ as coordinates on $H^{1}(\Gamma, \mathbb{C})$ (see Figure 11). The Casimirs are

$$
\begin{array}{lll}
C_{\alpha_{1}}=X_{1} X_{3} X_{7} B, & C_{\alpha_{2}}=\frac{B X_{2} X_{3} X_{4} X_{6} X_{7}}{X_{1} X_{5}}, & C_{\beta_{1}}=\frac{X_{2}}{A X_{1} X_{5}},
\end{array} \begin{array}{ll}
C_{\beta_{2}}=\frac{1}{A X_{7}} \\
C_{\gamma_{1}}=\frac{X_{5}}{B X_{1} X_{3}}, & C_{\gamma_{2}}=\frac{X_{6}}{B},
\end{array} C_{\delta_{1}}=\frac{A X_{1}}{X_{2} X_{6}}, \quad C_{\delta_{2}}=\frac{A X_{1} X_{5}}{X_{2} X_{3} X_{4} X_{6} X_{7}} .
$$

Since the Newton polygon $N$ has only one interior lattice point, the divisor $S=(p, q)$ consists of a


Figure 10: A square-octagon graph, its Newton polygon $N$ and normal fan $\Sigma$, with zig-zag paths and rays labeled.
single point. The Kasteleyn matrix is

$$
K=\left(\begin{array}{cccccccc}
1 & 1 & 0 & A z & 0 & 0 & 0 & 0 \\
1 & -X_{7} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{B w} \\
0 & 0 & \frac{X_{1} X_{5}}{X_{2} X_{3} X_{4} X_{6} X_{7}} & -1 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{X_{3}} & 0 & 0 & 1 & \frac{1}{X_{5}} & 0 & 0 \\
B w X_{1} & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{X_{1}}{X_{2}} & X_{6} & 1 \\
0 & 0 & 0 & 0 & \frac{1}{A z} & 0 & 1 & -1
\end{array}\right) .
$$

Let $P(z, w)=\operatorname{det} K$ and $\mathcal{C}=\overline{\{P(z, w)=0\}}$. The spectral transform is $\kappa_{\Gamma, \mathbf{w}}=(\mathcal{C}, S, \nu) \in \mathcal{S}_{N}$, where

$$
\begin{aligned}
p=- & \frac{X_{2} X_{4} X_{6}\left(X_{3} X_{5} X_{6} X_{7}\left(X_{1}^{2}\left(X_{4}+1\right)+X_{2} X_{4}\right)+X_{1} X_{2} X_{3}^{2} X_{4} X_{6}^{2} X_{7}^{2}+X_{1} X_{5}^{2}\right)}{A\left(X_{1} X_{5}+X_{2} X_{3} X_{4} X_{6} X_{7}\right)} \\
& \times \frac{1}{\left(X_{3} X_{4} X_{6} X_{7}\left(X_{1}^{2} X_{5}+X_{2}\left(X_{5}+1\right)\left(X_{6}+1\right)\right)+X_{1} X_{5}\left(X_{6}\left(X_{4}+X_{5}+1\right)+X_{5}+1\right)\right)} \\
q= & \frac{X_{5}\left(-X_{3} X_{4} X_{6} X_{7}\left(X_{1}^{2}+X_{2} X_{6}+X_{2}\right)-X_{1} X_{5}\left(X_{6}+1\right)\right)}{B X_{1} X_{3} X_{7}\left(X_{1} X_{5}\left(X_{4} X_{6}+X_{6}+1\right)+X_{2} X_{3} X_{4} X_{6}\left(X_{6}+1\right) X_{7}\right)}
\end{aligned}
$$

The table below lists the points at infinity for each of the zig-zag paths:


Figure 11: Labeling of the vertices and faces of $\Gamma$, and a cocycle and Kasteleyn sign, where $X_{i}=$ $[w t]\left(\left[\partial f_{i}\right]\right), A=[w t]([a]), B=[w t](b)$ and $U=\frac{X_{1} X_{5}}{X_{2} X_{3} X_{4} X_{6} X_{7}}$. The edges with no weight indicated have weight 1.

| Zig-zag path | Homology class | Basis $x_{1}, x_{2}$ | Point at infinity |
| :---: | :---: | :---: | :--- |
| $\alpha_{1}$ | $(0,1)$ | $(0,1),(-1,0)$ | $x_{1}=\frac{1}{C_{\alpha_{1}}}, x_{2}=0$ |
| $\alpha_{2}$ | $(-1,0)$ | $(-1,0),(0,-1)$ | $x_{1}=\frac{1}{C_{\alpha_{2}}}, x_{2}=0$ |
| $\beta_{1}$ |  | $\frac{1}{C_{\beta_{1}}}, x_{2}=0$ |  |
| $\beta_{2}$ | $(0,-1)$ | $(0,-1),(1,0)$ | $x_{1}=\frac{1}{C_{\beta_{2}}}, x_{2}=0$ |
| $\gamma_{1}$ |  | $\frac{1}{C_{\gamma_{1}}}, x_{2}=0$ |  |
| $\gamma_{2}$ | $(1,0)$ | $(1,0),(0,1)$ | $x_{1}=\frac{1}{C_{\gamma_{2}}}, x_{2}=0$ |
| $\delta_{1}$ |  | $x_{1}=\frac{1}{C_{\delta_{1}}}, x_{2}=0$ |  |
| $\delta_{2}$ |  |  |  |



Figure 12: The small polygons for the square-octagon graph.

The discrete Abel map $\mathbf{D}$ is given by $\mathbf{D}(\mathbf{w})=0$ and

$$
\begin{array}{ll}
\mathbf{D}\left(\mathrm{b}_{1}\right)=\frac{1}{2} D_{\gamma}+\frac{1}{2} D_{\delta}, & \mathbf{D}\left(\mathrm{b}_{2}\right)=\frac{1}{2} D_{\beta}+\frac{1}{2} D_{\gamma} \\
\mathbf{D}\left(\mathrm{b}_{3}\right)=\frac{1}{2}\left(-D_{\alpha}+D_{\beta}+D_{\gamma}+D_{\delta}\right), & \mathbf{D}\left(\mathrm{b}_{4}\right)=-D_{\alpha}+\frac{1}{2} D_{\beta}+D_{\gamma}+\frac{1}{2} D_{\delta} \\
\mathbf{D}\left(\mathrm{b}_{5}\right)=D_{\beta}, & \mathbf{D}\left(\mathrm{b}_{6}\right)=-\frac{1}{2} D_{\alpha}+D_{\beta}+\frac{1}{2} D_{\gamma} \\
\mathbf{D}\left(\mathrm{b}_{7}\right)=-\frac{1}{2} D_{\alpha}+\frac{1}{2} D_{\beta}+D_{\gamma}, & \mathbf{D}\left(\mathrm{b}_{8}\right)=-\frac{1}{2} D_{\alpha}+D_{\beta}+D_{\gamma}-\frac{1}{2} D_{\delta}
\end{array}
$$

We have $D_{N}=D_{\alpha}+D_{\beta}+D_{\gamma}+D_{\delta}$, using which we compute

$$
\begin{array}{ll}
Y_{\mathrm{b}_{1} \mathbf{w}}=\frac{1}{2} D_{\alpha}+D_{\beta}+D_{\gamma}+D_{\delta}, & Y_{\mathrm{b}_{2} \mathbf{w}}
\end{array}=\frac{1}{2} D_{\alpha}+D_{\beta}+D_{\gamma}+D_{\delta}, ~ Y_{\mathrm{b}_{4} \mathbf{w}}=D_{\beta}+\frac{3}{2} D_{\gamma}+D_{\delta}, ~ Y_{\mathrm{b}_{6} \mathbf{w}}=\frac{1}{2} D_{\alpha}+\frac{3}{2} D_{\beta}+D_{\gamma}+\frac{1}{2} D_{\delta}, ~ Y_{\beta}, \frac{3}{2} D_{\gamma}+D_{\delta}, \quad Y_{\mathrm{b}_{8} \mathbf{w}}=\frac{3}{2} D_{\beta}+\frac{3}{2} D_{\gamma}+\frac{1}{2} D_{\delta} .
$$

The corresponding small polygons are shown in Figure 12. Therefore, we have

$$
\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}=a_{(-1,-1)} z^{-1} w^{-1}+a_{(0,-1)} w^{-1}+a_{(1,-1)} z w^{-1}+a_{(-1,0)} z^{-1}+a_{(0,0)}+a_{(1,0)} z
$$

where the $a_{m}$ satisfy the system of equations $\mathbb{V}_{\mathrm{b}_{1} \mathbf{w}}$ that we now determine. We have the equation of type 1 :

$$
a_{(-1,-1)} p^{-1} q^{-1}+a_{(0,-1)} q^{-1}+a_{(1,-1)} p q^{-1}+a_{(-1,0)} p^{-1}+a_{(0,0)}+a_{(1,0)} p=0
$$

To find the zig-zag paths that contribute equations of type 2 , we compute (24). We have

$$
\begin{aligned}
-\left.D_{N}\right|_{\mathcal{C}} & =\nu\left(\alpha_{1}\right)+\nu\left(\alpha_{2}\right)+\nu\left(\beta_{1}\right)+\nu\left(\beta_{2}\right)+\nu\left(\gamma_{1}\right)+\nu\left(\gamma_{2}\right)+\nu\left(\delta_{1}\right)+\nu\left(\delta_{2}\right) \\
\mathbf{d}(\mathbf{w})-\mathbf{d}\left(\mathrm{b}_{1}\right) & =-\nu\left(\gamma_{1}\right)-\nu\left(\delta_{2}\right) \\
\left.\left\lfloor Y_{\mathrm{b}_{1} \mathbf{w}}\right\rfloor\right|_{\mathcal{C}} & =\nu\left(\beta_{1}\right)+\nu\left(\beta_{2}\right)+\nu\left(\gamma_{1}\right)+\nu\left(\gamma_{2}\right)+\nu\left(\delta_{1}\right)+\nu\left(\delta_{2}\right)
\end{aligned}
$$

using which we get that (24) is equal to $\nu\left(\beta_{1}\right)+\nu\left(\beta_{2}\right)+\nu\left(\gamma_{2}\right)+\nu\left(\delta_{1}\right)$, so we have four equations of type 2 , one for each of the zig-zag paths $\beta_{1}, \beta_{2}, \gamma_{2}, \delta_{1}$.

Therefore, we have 5 equations and 6 variables, so we are in the setting of Remark 3.6 where $\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}=\operatorname{det} \mathbb{V}_{\mathrm{b}_{1} \mathbf{w}}^{\chi}$. Computing the equations of type 2, we get

$$
\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}=\left|\begin{array}{cccccc}
z^{-1} w^{-1} & w^{-1} & z w^{-1} & z^{-1} & 1 & z \\
p^{-1} q^{-1} & q^{-1} & p q^{-1} & p^{-1} & 1 & p \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
1 & C_{\beta_{2}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1
\end{array}\right| .
$$

In like fashion, for $\mathrm{V}_{\mathrm{b}_{2} \mathbf{w}}$, we have an equation of type 1 and four equations of type 2 for the zig-zag paths $\beta_{1}, \gamma_{2}, \delta_{1}, \delta_{2}$. We compute

$$
\mathrm{V}_{\mathrm{b}_{2} \mathbf{w}}=\left|\begin{array}{cccccc}
z^{-1} w & w & z^{-1} & 1 & z^{-1} w^{-1} & w^{-1} \\
p^{-1} q & q & p^{-1} & 1 & p^{-1} q^{-1} & q^{-1} \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1 \\
0 & 0 & 0 & 0 & C_{\delta_{2}} & 1
\end{array}\right|
$$

We write the boundary of the face $f_{7}$ as the concatenation of the two wedges $W_{1}$ and $W_{2}$ represented by $\left(\mathbf{w}, \gamma_{1}\right)$ and ( $\mathrm{b}_{2}, \alpha_{1}$ ) respectively. We have

$$
r_{W_{1}}=\frac{\mathrm{V}_{\mathrm{b}_{2} \mathbf{w}}}{\mathrm{~V}_{\mathrm{b}_{1} \mathbf{w}}}\left(\nu\left(\gamma_{1}\right)\right)=\frac{\left|\begin{array}{cccccc}
C_{\gamma_{1}} & 0 & 1 & 0 & C_{\gamma_{1}}^{-1} & 0 \\
p^{-1} q & q & p^{-1} & 1 & p^{-1} q^{-1} & q^{-1} \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1 \\
0 & 0 & 0 & 0 & C_{\delta_{2}} & 1
\end{array}\right|}{\left|\begin{array}{cccccc}
C_{\gamma_{1}} & 0 & 1 & 0 & C_{\gamma_{1}}^{-1} & 0 \\
p^{-1} q^{-1} & q^{-1} & p q^{-1} & p^{-1} & 1 & p \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
1 & C_{\beta_{2}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1
\end{array}\right|},
$$

where to evaluate at $\nu\left(\gamma_{1}\right)$, we use the basis $x_{1}, x_{2}$ from table (35). Similarly, we compute

$$
r_{W_{2}}=-\frac{\mathrm{V}_{\mathrm{b}_{1} \mathbf{w}}}{\mathrm{~V}_{\mathrm{b}_{2} \mathbf{w}}}\left(\nu\left(\alpha_{1}\right)\right)=-\frac{\left|\begin{array}{cccccc}
0 & C_{\alpha_{1}}^{-1} & 0 & 1 & 0 & C_{\alpha_{1}} \\
p^{-1} q^{-1} & q^{-1} & p q^{-1} & p^{-1} & 1 & p \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
1 & C_{\beta_{2}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1
\end{array}\right|}{\left|\begin{array}{cccccc}
0 & C_{\alpha_{1}}^{-1} & 0 & 1 & 0 & C_{\alpha_{1}} \\
p^{-1} q & q & p^{-1} & 1 & p^{-1} q^{-1} & q^{-1} \\
1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\
C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\
0 & 0 & 0 & 0 & C_{\delta_{1}} & 1 \\
0 & 0 & 0 & 0 & C_{\delta_{2}} & 1
\end{array}\right|} .
$$

It can be verified using computer algebra that $X_{7}=r_{W_{1}} r_{W_{2}}$.

## 5 The small polygons

In the remaining sections, we prove the results stated in Section 3. In order to invert the spectral transform, we want to first reconstruct the $Q_{\mathrm{bw}}$, the entries of the w-column of the adjugate matrix, from the spectral data. To do this, we need to first find the Newton polygon of the $Q_{\mathrm{bw}}$, which we call the small polygons and denote by $N_{\mathrm{bw}}$. Explicitly, $N_{\mathrm{bw}}$ is the convex hull of homology classes of dimer covers of $\Gamma-\{b, \mathbf{w}\}$. However, it appears difficult to describe $N_{\mathrm{bw}}$ in a direct combinatorial way. Instead, we will re-express the problem in terms of toric geometry. The key to doing this is an extension of the Kasteleyn matrix, which is a map of trivial sheaves on T, to a map of locally free sheaves on a compactification of T . We are led to consider a stacky toric surface $X_{N}$ instead of the toric surface $X_{N}$, because such an extension does not exist on $X_{N}$ unless the polygon has only primitive sides.

The basics of stacky toric surfaces are recalled in detail in Appendix A.3. For the convenience of the reader we reproduce some notation.

Let $\Sigma$ be the normal fan of $N$. There is a stacky fan $\boldsymbol{\Sigma}=(\Sigma, \beta)$ where

$$
\begin{aligned}
\beta: \mathbb{Z}^{\Sigma(1)} & \rightarrow \mathrm{M}^{\vee}, \\
\delta_{\rho} & \mapsto\left|E_{\rho}\right| u_{\rho},
\end{aligned}
$$

where $u_{\rho}$ is the primitive normal to $E_{\rho}$. We identify the set of rays $\Sigma(1)$ of the fan $\Sigma$ with the components $D_{\rho}$ of the divisor at infinity

$$
\rho \leftrightarrow \tau_{\rho}=\mathbb{R}_{\geq 0} u_{\rho} .
$$

We assign to $\boldsymbol{\Sigma}$ a smooth toric Deligne-Mumford stack $X_{N}$, which contains the torus T as a dense open subset.

We consider the stack rather than the toric surface since we construct an extension of the Kasteleyn operator to a compactification of the torus T in Lemma 5.1. There is no such extension on the toric surface when the Newton polygon is not simple, but there is one on the stack.

### 5.1 Extension of the Kasteleyn operator

Define for each black vertex b the line bundle

$$
\mathcal{E}_{\mathrm{b}}:=\mathcal{O}_{x_{N}}\left(\mathbf{D}(\mathrm{~b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right)
$$

and for each white vertex w , the line bundle

$$
\mathcal{F}_{\mathrm{w}}:=\mathcal{O}_{x_{N}}(\mathbf{D}(\mathrm{w}))
$$

Let

$$
\mathcal{E}:=\bigoplus_{\mathrm{b} \in B} \mathcal{E}_{\mathrm{b}}, \quad \mathcal{F}:=\bigoplus_{\mathrm{w} \in W} \mathcal{F}_{\mathrm{w}}
$$

They are locally free sheaves of the same rank $\# B=\# W$ on $X_{N}$.
Proposition 5.1. The Kasteleyn operator $K$ extends to a map of locally free sheaves on $\mathcal{X}_{N}$ :

$$
\begin{equation*}
\widetilde{K}: \mathcal{E} \rightarrow \mathcal{F} \tag{36}
\end{equation*}
$$

Proof. By definition,

$$
K_{\mathrm{wb}}=\sum_{e \in E(\Gamma) \text { incident to } \mathrm{b}, \mathrm{w}} w t(e) \epsilon(e) \phi(e) .
$$

We need to show that for any edge $e$ with vertices $\mathrm{b}, \mathrm{w}$, the character $\phi(e)$ is a global section of

$$
\mathcal{H o m}_{X_{N}}\left(\mathcal{E}_{\mathrm{b}}, \mathcal{F}_{\mathrm{w}}\right) \cong \mathcal{O}_{X_{N}}\left(\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right)
$$

Let $m \in \mathrm{M}$ be such that $\phi(e)=\chi^{m}$ and let $D:=\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}$. By Proposition A.3, $\chi^{m}$ is a global section of the line bundle $\mathcal{O}_{X_{N}}(D)$ if and only if $m \in P_{D} \cap \mathrm{M}$. Using (52), this is equivalent to showing that for every edge $e=\mathrm{bw}$, we have

$$
\operatorname{div} \phi(e)+\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho} \geq 0
$$

where $\operatorname{div} \phi(e)=\sum_{\rho \in \Sigma(1)}\left\langle m, u_{\rho}\right\rangle D_{\rho}$ as in (48).
Let $\alpha, \beta$ be the zig-zag paths through $e$, with $\alpha \in Z_{\sigma}, \beta \in Z_{\rho}$. Then by Lemma 3.2, we have

$$
\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})=-\frac{1}{\left|E_{\sigma}\right|} D_{\sigma}-\frac{1}{\left|E_{\rho}\right|} D_{\rho}-\operatorname{div} \phi(e)
$$

This implies

$$
\begin{equation*}
\operatorname{div} \phi(e)+\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\tau \in \Sigma(1)} \sum_{\gamma \in Z_{\tau}: \mathrm{b} \in \gamma} \frac{1}{\left|E_{\tau}\right|} D_{\tau}=\sum_{\tau \in \Sigma(1)} \sum_{\substack{\gamma \in Z_{\tau}: \mathrm{b} \in \gamma \\ \gamma \neq \alpha, \beta}} \frac{1}{\left|E_{\tau}\right|} D_{\tau} \geq 0 . \tag{37}
\end{equation*}
$$

The small polygon $N_{\mathrm{bw}}$ is by definition the Newton polygon of $Q_{\mathrm{bw}}$. By Proposition A.3, this is equivalent to saying that $Q_{\mathrm{bw}}$ is a global section of a line bundle $\mathcal{O}_{x_{N}}\left(Y_{\mathrm{bw}}\right)$, where $Y_{\mathrm{bw}}$ is the divisor associated to $N_{\mathrm{bw}}$ by the correspondence (52). Now that we have shown that $K$ is a global section of $\mathcal{H o m}_{x_{N}}(\mathcal{E}, \mathcal{F})$, we can take exterior powers to find which line bundle $\mathcal{O}_{x_{N}}\left(Y_{\text {bw }}\right)$ the minor $Q_{\text {bw }}$ of $K$ is a global section of.

Taking the determinant of the map (36), we see that det $\widetilde{K}$ is a global section of the line bundle

$$
\begin{equation*}
\mathcal{H o m x}_{N}\left(\bigwedge_{\mathrm{b} \in B} \mathcal{E}_{\mathrm{b}}, \bigwedge_{\mathrm{w} \in W} \mathcal{F}_{\mathrm{w}}\right) \cong \mathcal{O}_{x_{N}}\left(\sum_{\mathrm{w} \in W} \mathbf{D}(\mathrm{w})-\sum_{\mathrm{b} \in B}\left(\mathbf{D}(\mathrm{~b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right)\right) \tag{38}
\end{equation*}
$$

Lemma 5.2. Let $D_{N}$ be the divisor associated to $N$ by the correspondence (52) between divisors and polygons. Then one has

$$
\begin{equation*}
\sum_{\mathrm{w} \in W} \mathbf{D}(\mathrm{w})-\sum_{\mathrm{b} \in B}\left(\mathbf{D}(\mathrm{~b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right)=D_{N} \tag{39}
\end{equation*}
$$

Therefore,

$$
\operatorname{det} \widetilde{K} \in H^{0}\left(X_{N}, \mathcal{O}_{x_{N}}\left(D_{N}\right)\right)
$$

Proof. Let $a_{\rho}$ be the coefficient of $D_{\rho}$ in $D_{N}$. Let $\left(i_{1}, i_{2}\right)$ be a vertex of $P$ contained in $E_{\rho}$ and let m be the associated extremal dimer cover. We pair up black and white vertices in the sum according to m :

$$
\sum_{e=\mathrm{bw} \in \mathrm{~m}}\left(\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right)
$$

Now we observe that if $e$ is not contained in any zig-zag path in $Z_{\rho}$, then $D_{\rho}$ does not appear in the summand, and if $e$ is contained in a zig-zag path associated to $E_{\rho}$, then $D_{\rho}$ appears twice but with opposite signs, modulo contributions from intersections of edges with $\gamma_{z}, \gamma_{w}$. Therefore, there is no net contribution to the coefficient of $D_{\rho}$ except for the intersections of edges in $m$ with $\gamma_{z}, \gamma_{w}$, which is the same as in

$$
-\sum_{e \in \mathrm{~m}} \operatorname{div} \phi(e)=-\operatorname{div} z^{i_{1}} w^{i_{2}}
$$

which is $a_{\rho}$. Comparing with (38), we see that (39) implies the second statement.
Now we consider the codimension 1 exterior power, where we remove $\{b, w\}$. Let $\widetilde{Q}$ be the adjugate matrix of $\widetilde{K}$. Set

$$
Y_{\mathrm{bw}}:=D_{N}-\mathbf{D}(\mathrm{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}
$$

Corollary 5.3. $\widetilde{Q}_{\mathrm{bw}} \in H^{0}\left(X_{N}, \mathcal{O}_{X_{N}}\left(Y_{\mathrm{bw}}\right)\right)$.
We therefore arrive at the definition of the small polygon $N_{\mathrm{bw}}$ given in Definition 3.3 by the correspondence (52).

### 5.2 Points at infinity

In this section, we prove that the points at infinity of $\mathcal{C}$ are as described in Section 2.7. We use the notations $U_{\Sigma} \subset \mathbb{C}^{\Sigma(1)}$ and the standard coordinates $\left(z_{\rho}\right)$ on $\mathbb{C}^{\Sigma(1)}$ from Appendix A.3. The toric variety $X_{N}$ is the quotient $U_{\Sigma} / H$, where $H$ is the kernel of the map $\left(\mathbb{C}^{\times}\right)^{\Sigma(1)} \rightarrow \mathrm{T}$ sending $\left(z_{\rho}\right)_{\rho \in \Sigma(1)}$ to $\left(\prod_{\rho} z_{\rho}^{\left\langle(1,0), u_{\rho}\right\rangle}, \prod_{\rho} z_{\rho}^{\left\langle(0,1), u_{\rho}\right\rangle}\right)$. There is a canonical map $\pi: U_{\Sigma} \rightarrow X_{N}$ given by sending $\left(z_{\rho}\right)$ to $H \cdot\left(z_{\rho}^{\left|E_{\rho}\right|}\right)$ which induces the coarse moduli space map $\pi: X_{N} \rightarrow X_{N}$. The spectral curve $\mathcal{C}$ is cut out by the section $P=P(z, w)$ of $\mathcal{O}_{X_{N}}\left(D_{N}\right)$. The pullback $\pi^{*} P$ defines a section of $\mathcal{O}_{X_{N}}\left(D_{N}\right)$ which is a $G$-invariant section of $\mathcal{O}_{U_{\Sigma}}$, so it vanishes on a $G$-invariant subvariety $\mathcal{C}_{U_{\Sigma}}$. Each point at infinity of $\mathcal{C}$ corresponds to a $G$-invariant set of points at infinity of $\mathcal{C}_{U_{\Sigma}}$, so we will determine the points at infinity of $\mathcal{C}$ from the points at infinity of $\mathcal{C}_{U_{\Sigma}}$. By Lemma $5.2, \pi^{*} P=\operatorname{det} \widetilde{K}$ so the points at infinity of $\mathcal{C}_{U_{\Sigma}}$ are obtained by setting $z_{\rho}=\operatorname{det} \widetilde{K}=0$ for $\rho \in \Sigma(1)$.

From (37) and Proposition A.3, for $e=\mathrm{bw}$, we get that $\phi(e)$ corresponds to the $G$-invariant section of $\mathcal{O}_{U_{\Sigma}}$ given by

$$
\begin{equation*}
\phi(e)=\prod_{\substack{\tau \in \Sigma(1)}} \prod_{\substack{\gamma \in Z_{\tau}: b \in \gamma \\ \gamma \neq \alpha, \beta}} z_{\tau} . \tag{40}
\end{equation*}
$$

The divisor $D_{\rho}$ in $X_{N}$ corresponds to $\left\{z_{\rho}=0\right\} \subset U_{\Sigma} . \phi(e)$ vanishes on $\left\{z_{\rho}=0\right\}$ precisely when there is a zig-zag path $\alpha \in Z_{\rho}$ such that b is contained in $\alpha$ but w is not contained in $\alpha$. This implies that when restricted to $\left\{z_{\rho}=0\right\}$, after reordering the black and white vertices, the extended Kasteleyn operator $\widetilde{K}$ takes a block-upper-triangular form

$$
\left(\begin{array}{ccccc}
\left.\widetilde{K}\right|_{\alpha_{1}} & & & & \\
& \left.\widetilde{K}\right|_{\alpha_{2}} & & & * \\
& & \ddots & & * \\
& & & \left.\widetilde{K}\right|_{\alpha_{n}} & \vdots \\
& & & & \left.\widetilde{K}\right|_{\Gamma-\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}}
\end{array}\right)
$$

where $Z_{\rho}=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\},\left.\widetilde{K}\right|_{\alpha_{i}}$ is the restriction of $\widetilde{K}$ to the black and white vertices in $\alpha_{i}$, and the *'s denote some possibly nonzero blocks whereas any block that has not been indicated is zero. In particular, the nonzero blocks $*$ are only in the last column (in these blocks, b is not in $\alpha$ but w is in $\alpha$ ). Note that the zig-zag paths $\alpha_{1}, \ldots, \alpha_{n}$ do not share any vertices because of minimiality since otherwise we would have a parallel bigon, so the blocks $\left.\widetilde{K}\right|_{\alpha_{1}}, \ldots,\left.\widetilde{K}\right|_{\alpha_{n}}$ do not overlap.

If $\alpha \in Z_{\rho}$ is $\mathrm{b}_{1} \rightarrow \mathrm{w}_{1} \rightarrow \mathrm{~b}_{2} \rightarrow \cdots \rightarrow \mathrm{w}_{d} \rightarrow \mathrm{~b}_{1}$, the determinant of the block $\left.\widetilde{K}\right|_{\alpha}$ is

$$
\begin{aligned}
\left.\operatorname{det} \widetilde{K}\right|_{\alpha} & =\operatorname{det}\left(\begin{array}{ccccc}
\widetilde{K}_{\mathrm{w}_{1} \mathrm{~b}_{1}} & & & & \widetilde{K}_{\mathrm{w}_{d} \mathrm{~b}_{1}} \\
\widetilde{K}_{\mathrm{w}_{1} \mathrm{~b}_{2}} & \widetilde{K}_{\mathrm{w}_{2} \mathrm{~b}_{2}} & & & \\
& \widetilde{K}_{\mathrm{w}_{2} \mathrm{~b}_{3}} & & & \\
& & \ddots & \ddots & \\
& & & \widetilde{K}_{\mathrm{w}_{d-1} \mathrm{~b}_{d}} & \widetilde{K}_{\mathrm{w}_{d} \mathrm{~b}_{d}}
\end{array}\right) \\
& =\prod_{i=1}^{d} \widetilde{K}_{\mathrm{w}_{i} \mathrm{~b}_{i}}-(-1)^{d} \prod_{i=1}^{d} \widetilde{K}_{\mathrm{w}_{i-1} \mathrm{~b}_{i}} \\
& =-\prod_{i=1}^{d}\left(w t\left(\mathrm{~b}_{i} \mathrm{w}_{i-1}\right) \epsilon\left(\mathrm{b}_{i} \mathrm{w}_{i-1}\right) \phi\left(\mathrm{b}_{i} \mathrm{w}_{i}\right)\right)\left(\prod_{i=1}^{d} \frac{\phi\left(\mathrm{~b}_{i} \mathrm{w}_{i-1}\right)}{\phi\left(\mathrm{b}_{i} \mathrm{w}_{i}\right)}-C_{\alpha}\right)
\end{aligned}
$$

where we have used the definition of the Kasteleyn matrix (see (11), (2). Plugging in (40) and using the fact that $\alpha$ intersects a zig-zag path $\beta \in Z_{\tau}\left\langle[\alpha], u_{\tau}\right\rangle$ times, we get

$$
\prod_{i=1}^{d} \frac{\phi\left(\mathrm{~b}_{i} \mathrm{w}_{i-1}\right)}{\phi\left(\mathrm{b}_{i} \mathrm{w}_{i}\right)}=\prod_{\tau \in \Sigma(1)} z_{\tau}^{-\left|E_{\tau}\right|\left\langle[\alpha], u_{\tau}\right\rangle}
$$

Therefore, the points at infinity of $\mathcal{C}_{U_{\Sigma}}$ are given by setting $z_{\rho}=0$ and $\prod_{\tau \in \Sigma(1)} z_{\tau}^{-\left|E_{\tau}\right|\left\langle[\alpha], u_{\tau}\right\rangle}=C_{\alpha}$. The point at infinity of $\mathcal{C}$ is the point obtained by applying $\pi$ to any of these points. From the definition of $\pi$, we get that $\prod_{\tau \in \Sigma(1)} z_{\tau}^{-\left|E_{\tau}\right|\left\langle[\alpha], u_{\tau}\right\rangle}=\pi^{*} \chi^{-[\alpha]}$, so the point at infinity of $\mathcal{C}$ is given by

$$
\begin{equation*}
\chi^{-[\alpha]}=C_{\alpha} \tag{41}
\end{equation*}
$$

## 6 Behaviour of the Laurent polynomial $Q_{\mathrm{bw}}(z, w)$ at infinity

We proved in Corollary 5.3 that the Laurent polynomial $Q_{\mathrm{bw}}(z, w)$ lies in the finite dimensional vector space $H^{0}\left(X_{N}, \mathcal{O}_{X_{N}}\left(Y_{\mathrm{bw}}\right)\right)$. We need some additional constraints on $Q_{\mathrm{bw}}(z, w)$ to determine it. Corollary 6.3 provides $g$ linear equations for the coefficients of $Q_{\mathrm{bw}}(z, w)$ coming from the vanishing of $Q_{\mathrm{bw}}(z, w)$ at the $g$ points of the divisor $S_{\mathrm{w}}$. We obtain additional equations from the behaviour of $Q_{\mathrm{bw}}(z, w)$ at the points at infinity of the spectral curve, which we study in this section.

Recall that $X_{N}$ is the toric surface associated to $N$ compactifying $T$. The restriction of the Kasteleyn operator to the open spectral curve $\mathcal{C}^{\circ}$ is a map of trivial sheaves:

$$
\left.K\right|_{\mathcal{C}^{\circ}}: \bigoplus_{\mathrm{b} \in B} \mathcal{O}_{\mathcal{C}^{\circ}} \longrightarrow \bigoplus_{w \in W} \mathcal{O}_{\mathcal{C}^{\circ}}
$$

Recall the correspondence between divisors $D$ and invertible sheaves with rational sections $(\mathcal{L}, s)$ : given a divisor $D$, the corresponding invertible sheaf $\mathcal{L}=\mathcal{O}_{\mathcal{C}}(D)$ is defined on an open $U$ by

$$
H^{0}\left(U, \mathcal{O}_{\mathcal{C}}(D)\right):=\left\{t \in K(\mathcal{C})^{\times}:\left.(\operatorname{div} t+D)\right|_{U} \geq 0\right\} \cup\{0\}
$$

with the obvious restriction maps, where $K(\mathcal{C})^{\times}$denotes the nonzero rational functions on $\mathcal{C}$. The rational section $s$ corresponds to the rational function 1 . On the other hand, given $(\mathcal{L}, s)$, we obtain
$D$ as the divisor div $s$. Moreover, there is a correspondence between rational functions $t$ on $\mathcal{C}$ and rational sections $\bar{t}$ of $\mathcal{L}$ given by $t \mapsto \bar{t}:=s t$. In particular,

$$
\begin{equation*}
\operatorname{div} \bar{t}=\operatorname{div} t+\operatorname{div} s=\operatorname{div} t+D \tag{42}
\end{equation*}
$$

and so $\bar{t}$ is regular if and only if $\operatorname{div} t+D \geq 0$.
A similar proof to Proposition 5.1 shows that the Kasteleyn matrix $K$, which is a matrix of rational functions on $\mathcal{C}$, defines a regular map $\bar{K}$ of locally free sheaves on $\mathcal{C}$ extending $\left.K\right|_{\mathcal{C}^{\circ}}$, providing an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{M} \rightarrow \bigoplus_{\mathrm{b} \in B} \mathcal{O}_{\mathcal{C}}\left(\mathbf{d}(\mathrm{b})-\sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha)\right) \xrightarrow{\bar{K}} \bigoplus_{\mathrm{w} \in W} \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathrm{w})) \rightarrow \mathcal{L} \rightarrow 0 \tag{43}
\end{equation*}
$$

where $\mathcal{M}$ and $\mathcal{L}$ are the kernel and cokernel of the map $\bar{K}$ respectively. When we say $\bar{K}$ is regular, we mean that each entry $\bar{K}_{\text {wb }}$ is a regular section of the corresponding $\mathcal{H}$ om line bundle

$$
\mathcal{H o m}_{\mathcal{O}_{\mathcal{C}}}\left(\mathcal{O}_{\mathcal{C}}\left(\mathbf{d}(\mathrm{b})-\sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha)\right), \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathrm{w}))\right)
$$

For generic dimer weights, $\Sigma$ is smooth, and $\mathcal{M}$ and $\mathcal{L}$ are line bundles (so $\bar{Q}$ has rank 1 ). Let $\bar{s}_{\mathrm{b}}$ and $\bar{s}_{\mathrm{w}}$ be sections of

$$
\mathcal{M}^{\vee} \otimes \mathcal{O}_{\mathcal{C}}\left(\mathbf{d}(\mathrm{b})-\sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha)\right) \text { and } \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathrm{w}))^{\vee}
$$

respectively, given by the b-entry of the kernel map and w-entry of the cokernel map respectively. Since $\bar{Q}$ has rank 1 , we have $\bar{Q}_{\mathrm{bw}}=\bar{s}_{\mathrm{b}} \bar{s}_{\mathrm{w}}$. Denote by $S_{\mathrm{b}}$ and $S_{\mathrm{w}}$ the effective divisors on the open spectral curve $\mathcal{C}^{\circ}$ given by vanishing of the b-row and w-column of $Q$ respectively, or equivalently, the vanishing of $\bar{s}_{\mathrm{b}}$ and $\bar{s}_{\mathrm{w}}$ respectively.

## Lemma 6.1.

$$
\begin{aligned}
& \operatorname{div}_{\mathcal{C}} \overline{\mathrm{s}}_{\mathrm{b}}=S_{\mathrm{b}}+\sum_{\alpha \in Z: \mathrm{b} \notin \alpha} \nu(\alpha), \\
& \operatorname{div}_{\mathcal{C}} \bar{s}_{\mathrm{w}}=S_{\mathrm{w}}
\end{aligned}
$$

Proof. By the definition, $\left.\left(\operatorname{div}_{\mathcal{C}} \bar{s}_{\mathrm{b}}\right)\right|_{\mathcal{C}^{\circ}}=S_{\mathrm{b}}$ and $\left.\left(\operatorname{div}_{\mathcal{C}} \bar{s}_{\mathrm{w}}\right)\right|_{\mathcal{C}^{\circ}}=S_{\mathrm{w}}$, so it only remains to find their orders of vanishing at infinity.

Let $U \subset \mathcal{C}$ be a neighbourhood of $\nu(\alpha)$ that does not contain any other point at infinity. Let $u$ be a local parameter in $U$ that vanishes to order 1 at $\nu(\alpha)$ and nowhere else. When restricted to $U$, each of the line bundles in the source and target of $\bar{K}$ in (43) is of the form $\mathcal{O}_{U}(k \nu(\alpha))$ for some $k \in \mathbb{Z}$. We trivialize $\mathcal{O}_{U}(k \nu(\alpha))$ as follows:

$$
\begin{aligned}
\mathcal{O}_{U}(k \nu(\alpha)) & \cong \\
f & \mapsto u^{k} f .
\end{aligned}
$$

Let us order the black and white vertices so that the vertices on $\alpha$ come first. Then in $U$, we have

$$
\bar{K}=\left(\begin{array}{cc}
K_{1} & B \\
u A & K_{2}
\end{array}\right)+O(u)
$$

where $K_{1}, K_{2}$ are the restrictions of $\bar{K}$ to $\alpha$ and $\Gamma-\alpha$ respectively. Since corank $\bar{K}=1$ and since we know corank $K_{1}>0$ from the computation of the determinant in Section 5.2 we have corank $K_{1}=1$ and that $K_{2}$ is invertible. Let $v \in \operatorname{ker} K_{1}$. Then,

$$
\operatorname{ker} \bar{K}=\left(v,-u K_{2}^{-1} A v\right)+O(u)
$$

If any entry in $v$ or $K_{2}^{-1} A v$ is 0 , then it means that some $\bar{s}_{\mathrm{b}}$ is identically 0 . Let $\bar{Q}$ denote the adjugate matrix of $\bar{K}$. Since $\bar{Q}$ has rank 1 , we have $\bar{Q}_{\mathrm{bw}}=\bar{s}_{\mathrm{b}} \bar{s}_{\mathrm{w}}=0$, so we will have $\bar{Q}_{\mathrm{bw}}=0$ for all $\mathrm{w} \in W$. On the other hand, $Q_{\mathrm{bw}}$ is the signed partition function for dimer covers of $\Gamma \backslash\{\mathrm{b}, \mathrm{w}\}$, so if we choose w such that bw is an edge of $\Gamma$ used in a dimer cover, then $Q_{\mathrm{bw}} \neq 0$ for generic dimer weights, a contradiction. Therefore, the entries of ker $\bar{K}$ are nonzero when $u \neq 0$, so $\bar{s}_{\mathrm{b}}$ has a simple zero at $\nu(\alpha)$ for all $\mathrm{b} \notin \alpha$ and has no zeroes or poles for $\mathrm{b} \in \alpha$.

Similarly, let $v^{\prime} \in \operatorname{ker} K_{1}^{*}$. We have

$$
\operatorname{ker} \bar{K}^{*}=\left(v^{\prime},-\left(K_{2}^{*}\right)^{-1} B v^{\prime}\right)+O(u)
$$

For generic dimer weights, none of the entries of $\left(K_{2}^{*}\right)^{-1} B v^{\prime}$ can vanish, so $\bar{s}_{\mathrm{w}}$ has no zeroes or poles at $\nu(\alpha)$.

Corollary 6.2. $\operatorname{div}_{\mathcal{C}} Q_{\mathrm{bw}}=S_{\mathrm{b}}+S_{\mathrm{w}}-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)$.
Proof. Let $\bar{Q}$ denote the adjugate matrix of $\bar{K}$. Since $\bar{Q}$ has rank 1, we have $\bar{Q}_{\mathrm{bw}}=\bar{s}_{\mathrm{b}} \bar{s}_{\mathrm{w}}$, so that

$$
\operatorname{div}_{\mathcal{C}} \bar{Q}_{\mathrm{bw}}=S_{\mathrm{b}}+S_{\mathrm{w}}+\sum_{\alpha \in Z: \mathrm{b} \notin \alpha} \nu(\alpha)
$$

A computation similar to Corollary 5.3 shows that

$$
\bar{Q}_{\mathrm{bw}} \in H^{0}\left(\mathcal{C}, \mathcal{O}_{\mathcal{C}}\left(-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha)\right)\right)
$$

$Q_{\mathrm{bw}}$ is the rational function corresponding to the rational section $\bar{Q}_{\mathrm{bw}}$. Therefore, using (42), we have

$$
\begin{aligned}
\operatorname{div}_{\mathcal{C}} Q_{\mathrm{bw}} & =\operatorname{div}_{\mathcal{C}} \bar{Q}_{\mathrm{bw}}-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha) \\
& =S_{\mathrm{b}}+S_{\mathrm{w}}-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)
\end{aligned}
$$

Corollary 6.3. We have for all $\mathrm{b} \in B, \mathrm{w} \in W$, $\operatorname{deg} S_{\mathrm{b}}=\operatorname{deg} S_{\mathrm{w}}=g$, where $g$ is the genus of $\mathcal{C}$.
Proof. Let $\omega_{*}$ denote the canonical divisor of $*$. We have $\omega_{X_{N}}=-\sum_{\rho \in \Sigma(1)} D_{\rho}$ (CLS11, Theorem 8.2.3]). By the adjunction formula, we get $\omega_{\mathcal{C}}=\left.D_{N}\right|_{\mathcal{C}}-\sum_{\alpha \in Z} \nu(\alpha)$. Since $Q_{\mathrm{bw}}$ is a rational function on $\mathcal{C}$, we have $\operatorname{deg} \operatorname{div}_{\mathcal{C}} Q_{\mathrm{bw}}=0$. Since $\operatorname{deg}(\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b}))=-2$ and $\operatorname{deg} \omega_{\mathcal{C}}=2 g-2$, we get $\operatorname{deg}\left(S_{\mathrm{b}}+S_{\mathrm{w}}\right)=2 g$. By symmetry under interchanging $B$ and $W$, we get $\operatorname{deg} S_{\mathrm{b}}=\operatorname{deg} S_{\mathrm{w}}=g$.

Recall that the number $g$ is also equal to the number of interior lattice points in $N$ for generic $\mathcal{C} \in\left|D_{N}\right|$.

Proposition 6.4. The line bundle $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathcal{C}}\left(S_{\mathrm{w}}+\mathbf{d}(\mathrm{w})\right)$ for any $\mathrm{w} \in W$. It has degree $g-1$.
Proof. By Lemma 6.1 $\bar{s}_{\mathrm{w}}$ is a section of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathrm{w}))^{\vee}$ with divisor $S_{\mathrm{w}}$. Therefore, we must have

$$
\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathrm{w}))^{\vee} \cong \mathcal{O}_{\mathcal{C}}\left(S_{\mathrm{w}}\right)
$$

which implies $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}\left(S_{\mathrm{w}}+\mathbf{d}(\mathrm{w})\right)$. Since $\operatorname{deg} S_{\mathrm{w}}=g$ and $\operatorname{deg} \mathbf{d}(\mathrm{w})=-1$, we get $\operatorname{deg} \mathcal{L}=g-1$.

## 7 Equations for the Laurent polynomial $Q_{b w}$

Since $Q_{\text {bw }}$ has Newton polygon $N_{\text {bw }}$, we have

$$
Q_{\mathrm{bw}}=\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m}
$$

for some $a_{m} \in \mathbb{C}$. We know that $Q_{\mathbf{b w}}$ vanishes on $S_{\mathbf{w}}$, which gives $g$ linear equations among the $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$. However these $g$ linear equations are not usually sufficient to determine $Q_{\mathrm{bw}}$, so we need to find some additional equations. These additional equations will come from the vanishing of $Q_{\mathrm{bw}}$ at the points at infinity.

### 7.1 Additional linear equations for $Q_{b w}$

The fact that the Newton polygon of $Q_{\mathrm{bw}}$ is the small polygon $N_{\mathrm{bw}}$ imposes certain inequalities on the order of vanishing of $Q_{\mathrm{bw}}$ at points at infinity of $\mathcal{C}$. Corollary 6.2 imposes additional constraints that are linear equations in the coefficients of $Q_{\mathbf{b w}}$. Inverting this linear system gives $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$ and therefore $Q_{\mathrm{bw}}$.

We now give the precise statement. For a $\mathbb{Q}$-divisor $D=\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$, we define a ( $\mathbb{Z}$ - ) divisor $\lfloor D\rfloor:=\sum_{\rho \in \Sigma(1)}\left\lfloor b_{\rho}\right\rfloor D_{\rho}$, where $\lfloor x\rfloor$ is the largest integer $n$ such that $n \leq x$. It is the pushforward of $D$ by the canonical projection $X_{N} \rightarrow X_{N}$.

Proposition 7.1. The extra linear equations for $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$ from vanishing of $Q_{\mathrm{bw}}$ at points at infinity correspond to the points in

$$
\begin{equation*}
-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathbf{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)+\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}} \tag{44}
\end{equation*}
$$

Proof. A generic Laurent polynomial $F$ of the form $\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m}$ has order of vanishing

$$
\left.\operatorname{div}_{\mathcal{C}} F\right|_{\mathcal{C}} \geq-\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}
$$

at the points at infinity of $\mathcal{C}$. From Corollary 6.2, we have that $\operatorname{div}_{\mathcal{C}} Q_{\mathrm{bw}}=S_{\mathrm{b}}+S_{\mathbf{w}}-\left.D_{N}\right|_{\mathcal{C}}+$ $\mathbf{d}(\mathbf{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)$. The discrepancy provides the extra equations.

Now we describe these extra linear equations explicitly. Suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes a linear equation. We extend $[\alpha]$ to a basis $\left([\alpha]=x_{1}, x_{2}\right)$ of M such that $\left\langle x_{2}, u_{\rho}\right\rangle=1$, so that for any $m \in \mathrm{M}$, we can write

$$
\chi^{m}=x_{1}^{b_{m}} x_{2}^{c_{m}}, \quad b_{m}, c_{m} \in \mathbb{Z}
$$

Let $N_{\mathrm{bw}}^{\rho}$ be the set of lattice points in $N_{\mathrm{bw}}$ closest to the edge $E_{\rho}$ of $N$ i.e., the set of points in $N_{\text {bw }}$ that minimize the functional $\left\langle *, u_{\rho}\right\rangle$.

Proposition 7.2. Suppose $Q_{\mathbf{b w}}=\sum_{m \in N_{\mathrm{bw}} \cap \mathrm{M}} a_{m} \chi^{m}$ and suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes a linear equation. Then, the linear equation given by $\alpha$ is:

$$
\sum_{m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}} a_{m} C_{\alpha}^{-b_{m}}=0 .
$$

Proof. The affine open variety in $X_{N}$ corresponding to the cone $\rho$ is

$$
U_{\rho}=\operatorname{Spec} \mathbb{C}\left[x_{1}^{ \pm 1}, x_{2}\right] \cong \mathbb{C}^{\times} \times \mathbb{C}
$$

and $D_{\rho} \cap U_{\rho}$ is defined by $x_{2}=0$.
A generic curve $\mathcal{C}$ meets $D_{\rho}$ transversely at $\nu(\alpha)$, and therefore we may take $x_{2}$ as a uniformizer of the local ring $\mathcal{O}_{\mathcal{C}, \nu(\alpha)}$ at $\nu(\alpha)$. For each $m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}$, we have

$$
\chi^{m}=x_{1}^{b_{m}} x_{2}^{p}, \quad b_{\gamma}, p \in \mathbb{Z}
$$

where $p$ is the same for all of them, and is the coefficient of $\nu(\alpha)$ in $-\left.\left[E_{\mathbf{b} w}\right]\right|_{\mathcal{C}}$. Then using $x_{1}^{-1}=C_{\alpha}$ at $\nu(\alpha)$, we have

$$
\begin{equation*}
Q_{\mathrm{bw}}=\left(\sum_{m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}} a_{m} C_{\alpha}^{-b_{m}}\right) x_{2}^{p}+O\left(x_{2}^{p+1}\right) \tag{45}
\end{equation*}
$$

Since $\alpha$ contributes a linear equation, (45) must vanish at order $x_{2}^{p}$, so $\sum_{m \in N_{\mathrm{bw}}^{\rho} \cap \mathrm{M}} a_{m} C_{\alpha}^{-b_{m}}=0$.

### 7.2 The system of linear equations $\mathbb{V}_{b w}$

Recall from Section 3 the system of linear equations $\mathbb{V}_{\text {bw }}$. These are linear equations in the variables $\left(a_{m}\right)_{m \in N_{\mathrm{bw}} \cap \mathrm{M}}$. Recall also that the matrix $\mathbb{V}_{\mathrm{bw}}$ is defined such that these equations are given by

$$
\mathbb{V}_{\mathrm{bw}}\left(a_{m}\right)=0
$$

It is not necessarily a square matrix. However, we have:
Proposition 7.3. For generic spectral data, $Q_{\mathrm{bw}}$ is the unique solution of the linear system of equations $\mathbb{V}_{\text {bw }}$ modulo scaling.
Remark 7.4. 1. While the definition of $\mathbb{V}_{\text {bw }}$ makes sense for all $\mathrm{w} \in W$, Proposition 7.3 only holds when $\mathbf{w}=\mathbf{w}$ since $\left(p_{i}, q_{i}\right)_{i=1}^{g}$ depends on $\mathbf{w}$.
2. For generic spectral data, the equations (23) are linearly independent, but the equations (25) may not be.
The rest of this section is devoted to the proof of Proposition 7.3. Consider following the exact sequence on $X_{N}$, obtained by tensoring the closed embedding exact sequence of $i: \mathcal{C} \hookrightarrow X_{N}$ by $\mathcal{O}_{X_{N}}\left(\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right)$.

$$
0 \rightarrow \mathcal{O}_{X_{N}}\left(\left\lfloor Y_{\mathrm{bw}}\right\rfloor-D_{N}\right) \rightarrow \mathcal{O}_{X_{N}}\left(\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right) \rightarrow i_{*} \mathcal{O}_{\mathcal{C}}\left(\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}\right) \rightarrow 0
$$

The following is a portion of the long exact sequence of cohomology.

$$
\begin{equation*}
0 \rightarrow H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor-D_{N}\right) \rightarrow H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right) \rightarrow H^{0}\left(\mathcal{C},\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}\right) \tag{46}
\end{equation*}
$$

We need the following technical lemma.

Lemma 7.5. The restriction map $H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right) \rightarrow H^{0}\left(\mathcal{C},\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}\right)$ is injective.
Proof. If $\chi^{m} \in H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor-D_{N}\right)$, then $\operatorname{div} \chi^{m}+\left\lfloor Y_{\mathrm{bw}}\right\rfloor-D_{N} \geq 0$. This implies that

$$
\begin{equation*}
\operatorname{div} \chi^{m}+Y_{\mathrm{bw}}-D_{N}=\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}}\left\langle m, u_{\rho}\right\rangle \frac{D_{\rho}}{\left|E_{\rho}\right|}-\mathbf{D}(\mathbf{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho} \geq 0 \tag{47}
\end{equation*}
$$

Let $\gamma$ be a cycle in $\mathbb{T}$ with homology class $m$. The total number of signed intersections of $\gamma$ with all zig-zag paths is zero. This number is the sum of the coefficients of $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}}\left\langle m, u_{\rho}\right\rangle \frac{D_{\rho}}{\left|E_{\rho}\right|}$. Let $\mathrm{w}^{\prime}$ be any white vertex adjacent to b . Then we have

$$
-\mathbf{D}(\mathbf{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}=\left(\mathbf{D}\left(\mathrm{w}^{\prime}\right)-\mathbf{D}(\mathbf{w})\right)-\sum_{\substack{\rho \in \Sigma(1)}} \sum_{\substack{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha \\ \mathrm{bw} \\ \mathrm{~b} \neq \alpha}} \frac{1}{\left|E_{\rho}\right|} D_{\rho} .
$$

The sum of the coefficients of $\mathbf{D}\left(\mathrm{w}^{\prime}\right)-\mathbf{D}(\mathbf{w})$ is the signed number of intersections with zigzag paths of any path in $R$ from $\mathbf{w}$ to $\mathrm{w}^{\prime}$, which is also 0 . Since the coefficients of the last term $-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}$ are strictly negative, the sum in (47) cannot be non-negative. Therefore, $H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}^{\mathrm{bw}} \not \ddagger \alpha\right)-D_{N}\right)=0$, which by (46) means that the map $H^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right) \rightarrow$ $H^{0}\left(\mathcal{C},\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}\right)$ is injective.
Proof of Proposition 7.3. 1. Existence: By Theorem 7.3 of [GK13, the map $\kappa_{\Gamma, \mathbf{w}}$ is dominant. So a generic spectral data is in the image of $\kappa_{\Gamma, \mathbf{w}}$. For such a spectral data, $Q_{\mathrm{b} \mathbf{w}}$ satisfies:
(a) The system of equations (23) because, by definition of the spectral transform, $Q_{\mathrm{bw}}$ vanishes at the points of the divisor $S=\sum_{i=1}^{g}\left(p_{i}, q_{i}\right)$.
(b) The equations (25) by Proposition 7.2
2. Uniqueness: Suppose $V_{\mathrm{bw}}$ is a solution of $\mathbb{V}_{\mathrm{bw}}$. Since $V_{\mathrm{bw}}$ has Newton polygon $N_{\mathrm{bw}}$, we have $\left.\operatorname{div}_{\mathcal{C}} F\right|_{\mathcal{C}} \geq-\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}$ as in the proof of Proposition 7.1. The additional equations in Proposition 7.1 then imply that

$$
\operatorname{div}_{\mathcal{C}} V_{\mathrm{bw}} \geq S+D
$$

where $D:=-\left.D_{N}\right|_{\mathcal{C}}+\mathbf{d}(\mathbf{w})-\mathbf{d}(\mathrm{b})+\sum_{\alpha \in Z} \nu(\alpha)$ satisfies $\operatorname{deg} D=-2 g$. Therefore, $\left.V_{\mathrm{bw}}\right|_{\mathcal{C}}$ can be identified with a section of $\mathcal{O}_{\mathcal{C}}(-D)$ vanishing at the points of $S$. Let $\omega_{\mathcal{C}}$ denote the canonical divisor of $\mathcal{C}$ as in Section 6. By the Riemann-Roch theorem,

$$
h^{0}\left(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)-h^{1}\left(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)\right)=\operatorname{deg}(-D)-g+1=g+1\right.
$$

By Serre duality, $h^{1}\left(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)\right)=h^{0}\left(\mathcal{C}, \omega_{\mathcal{C}}(D)\right)$, which equals 0 since $\omega_{\mathcal{C}}(D)$ has negative degree -2 . For generic $S$ that avoids the base locus of $\mathcal{O}_{\mathcal{C}}(-D)$, the requirement that the section of $\mathcal{O}_{\mathcal{C}}(-D)$ corresponding to $V_{\mathrm{bw}}$ vanishes at each of the $g$ points of $S$ imposes $g$ independent conditions, and therefore determines $\left.V_{\mathrm{bw}}\right|_{\mathcal{C}}$ uniquely up to multiplication by a nonzero complex number. By Lemma [7.5, $V_{\mathrm{bw}}$ is unique up to multiplication by a nonzero complex number.

Remark 7.6. It is easy to see using Riemann-Roch that the number of equations in $\mathbb{V}_{\mathrm{bw}}$ is equal to $h^{0}\left(\mathcal{C},\left.\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right|_{\mathcal{C}}\right)-1$. On the other hand, the number of variables is $h^{0}\left(X_{N},\left\lfloor Y_{\mathrm{bw}}\right\rfloor\right)$. However, the map in Lemma 7.5 is not necessarily an isomorphism (there may be sections on the curve that are not restrictions of sections on the surface), so we only have the inequality

$$
\# \text { equations in } \mathbb{V}_{\mathrm{bw}} \geq \# \text { variables }-1
$$

## A Toric geometry

In A. 1 and A.2 we give a brief background on toric varieties; further details can be found in the books Ful93] and CLS11.

## A. 1 Toric varieties

A toric variety $X$ over $\mathbb{C}$ is an algebraic variety containing the complex algebraic torus $\mathrm{T} \cong\left(\mathbb{C}^{\times}\right)^{n}$ as a Zariski open subset, such that the action of T on itself extends to an action of T on $X$.

Let M be a lattice, and let $\mathrm{M}^{\vee}:=\operatorname{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ denote the dual lattice. Let $\mathrm{T}:=\mathrm{M}^{\vee} \otimes \mathbb{C}^{\times}=$ $\operatorname{Hom}\left(\mathrm{M}, \mathbb{C}^{\times}\right)$be the complex algebraic torus with the lattice of characters M . We denote by $\chi^{m}: \mathrm{T} \rightarrow \mathbb{C}^{\times}$the character associated to $m \in \mathrm{M}$. Let $\langle *, *\rangle$ be the pairing between M and $\mathrm{M}^{\vee}$. In our case $\mathrm{M}=H_{1}(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^{2}$, so $\mathrm{M}^{\vee}=H^{1}(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^{2}$ and $\mathrm{T}=H^{1}\left(\mathbb{T}, \mathbb{C}^{\times}\right) \cong\left(\mathbb{C}^{\times}\right)^{2}$. We have $\chi^{(i, j)}(z, w)=z^{i} w^{j}$.

A fan $\Sigma$ is a collection of cones in the real vector space $M_{\mathbb{R}}^{\vee}:=M^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$, which is just the Lie algebra of the real torus $T(\mathbb{R})$, such that

1. Each face of a cone $\sigma \in \Sigma$ is also in $\Sigma$.
2. The intersection of two cones $\sigma_{1}, \sigma_{2} \in \Sigma$ is a face of each of them.

Each cone $\sigma \in \Sigma$ gives rise to an affine toric variety

$$
\mathrm{U}_{\sigma}=\operatorname{Spec} \mathbb{C}\left[S_{\sigma}\right]
$$

where $S_{\sigma}=\sigma^{\vee} \cap \mathrm{M}$ is a semigroup, $\sigma^{\vee}$ is the cone dual to $\sigma$, and $\mathbb{C}\left[S_{\sigma}\right]$ is its semigroup algebra:

$$
\mathbb{C}\left[S_{\sigma}\right]=\left\{\sum_{m \in S_{\sigma}} c_{m} \chi^{m}: c_{m} \in \mathbb{C}, c_{m}=0 \text { for all but finitely many } m \in S_{\sigma}\right\}
$$

If $\tau \subset \sigma$, then $\mathrm{U}_{\tau}$ is an open subset of $\mathrm{U}_{\sigma}$. Gluing the affine toric varieties $\mathrm{U}_{\sigma_{1}}, \mathrm{U}_{\sigma_{2}}$ along $\mathrm{U}_{\sigma_{1} \cap \sigma_{2}}$ for all cones $\sigma_{1}, \sigma_{2} \in \Sigma$, we get the toric variety $X_{\Sigma}$ associated to $\Sigma$.

In particular, if $\sigma=\{0\}$, then $S_{\sigma}=\mathrm{M}$, so $\mathbb{C}\left[S_{\sigma}\right]=\mathbb{C}[\mathrm{M}]$ and $\mathrm{U}_{\sigma}=\mathrm{T}$. So $X_{\Sigma}$ contains T .
We define the action $\mathrm{T} \times \mathrm{U}_{\sigma} \longrightarrow \mathrm{U}_{\sigma}$ via the dual map of the algebras of functions:

$$
\begin{aligned}
\mathbb{C}\left[S_{\sigma}\right] & \longrightarrow \mathbb{C}[\mathrm{M}] \otimes \mathbb{C}\left[S_{\sigma}\right] \\
\chi^{m} & \longmapsto \chi^{m} \otimes \chi^{m}
\end{aligned}
$$

When $\sigma=\{0\}$, this is the action of T on itself. The action of T on $\mathrm{U}_{\sigma}$ is compatible with the gluing, and therefore gives an action of T on $X_{\Sigma}$.

We denote by $\Sigma(r)$ the set of $r$-dimensional cones of $\Sigma$. There is an inclusion-reversing bijection between T-orbit closures in $X_{\Sigma}$ and cones in $\Sigma$. Under this bijection, each ray $\rho \in \Sigma(1)$ corresponds to a T-invariant divisor $D_{\rho}$. Let $u_{\rho}$ be the primitive vector generating $\rho$. Then, the (Weil) divisor of the character $\chi^{m}$ is

$$
\begin{equation*}
\operatorname{div} \chi^{m}=\sum_{\rho \in \Sigma(1)}\left\langle m, u_{\rho}\right\rangle D_{\rho} \tag{48}
\end{equation*}
$$

The following fundamental exact sequence computes the class group of Weil divisors of $X_{\Sigma}$ :

$$
\begin{align*}
0 \rightarrow \mathrm{M} & \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow \mathrm{Cl}\left(X_{\Sigma}\right) \rightarrow 0  \tag{49}\\
& m \mapsto\left(\left\langle m, u_{\rho}\right\rangle\right)_{\rho \in \Sigma(1)}
\end{align*}
$$

## A. 2 Polygons and projective toric surfaces

Given a convex integral polygon $N$ in the plane $\mathrm{M}_{\mathbb{R}}:=\mathrm{M} \otimes_{\mathbb{Z}} \mathbb{R}$, we construct the normal fan $\Sigma$ of $N$ as follows:

1. $\Sigma(0)=\{0\}$.
2. For each edge $E_{\rho}$ of $N$, let $u_{\rho} \in \mathrm{M}^{\vee}$ be the primitive inward normal vector to $E_{\rho}$, providing an element of $\Sigma(1)$ given by the ray spanned by $u_{\rho}$.
3. For each vertex $v$ of $N$, we get an element of $\Sigma(2)$ by taking the convex hull of the two rays in $\Sigma(1)$ associated to the two edges incident to $v$ in $N$.

The normal fan $\Sigma$ gives rise to a toric surface denoted below by $X_{N}$. The orbit-cone correspondence assigns to each edge $E_{\rho}$ of $N$ a divisor $D_{\rho} \cong \mathbb{P}^{1}$. These divisors intersect according to the combinatorics of $N$. Their union is the divisor at infinity $X_{N}-\mathrm{T}$.

In fact the polygon $N$ determines a pair $\left(X_{N}, D_{N}\right)$, where $D_{N}$ is an ample divisor at infinity:

$$
D_{N}:=\sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}
$$

where $a_{\rho}$ is such that the edge $E_{\rho}$ of $N$ is contained in the line $\left\{m \in \mathrm{M} \otimes \mathbb{R}:\left\langle m, u_{\rho}\right\rangle=-a_{\rho}\right\}$.
The linear system of hyperplane sections $\left|D_{N}\right|$ has the following properties:

1. $H^{0}\left(X_{N}, \mathcal{O}_{X_{N}}\left(D_{N}\right)\right) \cong \bigoplus_{m \in N \cap M} \mathbb{C} \cdot \chi^{m}$.
2. The genus of a generic curve $C$ in $\left|D_{N}\right|$ is the number of interior lattice points of $N$.
3. Curves in $\left|D_{N}\right|$ intersect the divisor $D_{\rho}$ with multiplicity $\left|E_{\rho}\right|$ (the number of primitive vectors in $E_{\rho}$ ).

## A. 3 Toric stacks

We follow BH09, Section 2]. Given a convex integral polygon $N \subset \mathrm{M}_{\mathbb{R}}$, we define a stacky fan $\boldsymbol{\Sigma}$ as the following data:

1. The normal fan $\Sigma$ of $N$, defined above.
2. For each ray $\rho \in \Sigma(1)$, the vector $\left|E_{\rho}\right| u_{\rho}$ generating the ray $\rho$.

We define a fan $\widetilde{\Sigma} \subset \mathbb{R}^{\Sigma(1)}$ as follows: for $\sigma \in \Sigma$, we define $\widetilde{\sigma} \in \widetilde{\Sigma}$ by

$$
\widetilde{\sigma}=\operatorname{cone}\left(e_{\rho}: \rho \in \sigma(1)\right) \subset \mathbb{R}^{\Sigma(1)},
$$

where $\left\{e_{\rho}\right\}$ is the standard basis in $\rho$ in $\mathbb{R}^{\Sigma(1)}$, and $\sigma(1)$ denotes the rays of $\Sigma$ incident to $\sigma$. Then $\widetilde{\Sigma}$ is the fan generated by the cones $\widetilde{\sigma}$ and their faces.

Let $U_{\Sigma}$ be the toric variety of the fan $\widetilde{\Sigma}$. It is of the form $\mathbb{C}^{\Sigma(1)}-($ closed codimension 2 subset $)$. Consider the following map, modifying the map (49) for polygons $N$ with a non-primitive side:

$$
\begin{aligned}
\beta: \mathrm{M} & \rightarrow \mathbb{Z}^{\Sigma(1)} \\
m & \mapsto\left(\left|E_{\rho}\right|\left\langle m, u_{\rho}\right\rangle\right)_{\rho} .
\end{aligned}
$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}}\left(*, \mathbb{C}^{\times}\right)$, we get a surjective map $\left(\mathbb{C}^{\times}\right)^{\Sigma(1)} \rightarrow$ T. Denote by $G$ its kernel. So there is an exact sequence

$$
\begin{equation*}
1 \rightarrow G \rightarrow\left(\mathbb{C}^{\times}\right)^{\Sigma(1)} \rightarrow \mathrm{T} \rightarrow 1 \tag{50}
\end{equation*}
$$

So $G$ is a subgroup of the torus $\left(\mathbb{C}^{\times}\right)^{\Sigma(1)}$ of the toric variety $U_{\Sigma}$. Therefore, $G$ acts on $U_{\Sigma}$.
Explicitly, $\lambda=\left(\lambda_{\rho}\right) \in\left(\mathbb{C}^{\times}\right)^{\Sigma(1)}$ is in $G$ if and only if

$$
\begin{equation*}
\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{\left|E_{\rho}\right|\left\langle m, u_{\rho}\right\rangle}=1 \tag{51}
\end{equation*}
$$

for all $m \in \mathrm{M}$. Let $z=\left(z_{\rho}\right) \in \mathbb{C}^{\Sigma(1)}$ denote the standard coordinates on $\mathbb{C}^{\Sigma(1)}$. The action of $G$ on $U_{\Sigma}$ is $\lambda \cdot z=\left(\lambda_{\rho} z_{\rho}\right)$.

Definition A.1. The toric stack $X_{N}$ is the smooth Deligne-Mumford stack $\left[U_{\Sigma} / G\right]$.

## A. 4 Example: a stacky $\mathbb{P}^{2}$.

Consider the polygon $N$ given by the convex-hull of $\{(0,0),(2,0),(0,2)\}$. The rays of its normal fan $\Sigma$ are generated by $u_{1}=(1,0), u_{2}=(0,1), u_{3}=(-1,-1)$ with $\left|E_{1}\right|=\left|E_{2}\right|=\left|E_{3}\right|=2$. The fan $\widetilde{\Sigma} \subset \mathbb{R}^{3}$ is generated by the cones

$$
\widetilde{\sigma}_{1}=\operatorname{cone}\left(e_{2}, e_{3}\right), \quad \widetilde{\sigma}_{2}=\operatorname{cone}\left(e_{1}, e_{3}\right), \quad \widetilde{\sigma}_{3}=\operatorname{cone}\left(e_{1}, e_{2}\right)
$$

and their faces, where $\left\{e_{i}\right\}$ is the standard basis of $\mathbb{R}^{3}$. These cones define affine varieties

$$
U_{1}=\operatorname{Spec} \mathbb{C}\left[X_{1}^{ \pm 1}, X_{2}, X_{3}\right], \quad U_{2}=\operatorname{Spec} \mathbb{C}\left[X_{1}, X_{2}^{ \pm 1}, X_{3}\right], \quad U_{3}=\operatorname{Spec} \mathbb{C}\left[X_{1}, X_{2}, X_{3}^{ \pm 1}\right]
$$

respectively. The face $\widetilde{\sigma}_{12}:=\widetilde{\sigma}_{1} \cap \widetilde{\sigma}_{2}=\operatorname{cone}\left(e_{3}\right)$ defines the affine variety $U_{12}=\operatorname{Spec} \mathbb{C}\left[X_{1}^{ \pm 1}, X_{2}^{ \pm 1}, X_{3}\right]$, identified with $U_{1} \cap U_{2}$. Similarly, we define $U_{23}$ and $U_{13}$. Gluing $U_{i}$ and $U_{j}$ along the $U_{i j}$ for all $i, j$, we see that the toric variety $U_{\Sigma}$ of $\widetilde{\Sigma}$ is $\mathbb{C}^{3}-0$. The map $M \rightarrow \mathbb{Z}^{\Sigma(1)}$ is

$$
\begin{aligned}
\mathbb{Z}^{2} & \rightarrow \mathbb{Z}^{3} \\
(1,0) & \mapsto(2,0,-2) \\
(0,1) & \mapsto(0,2,-2)
\end{aligned}
$$

The group $G$ is the kernel of

$$
\begin{aligned}
\left(\mathbb{C}^{\times}\right)^{3} & \rightarrow\left(\mathbb{C}^{\times}\right)^{2} \\
\left(t_{1}, t_{2}, t_{3}\right) & \mapsto\left(\left(\frac{t_{1}}{t_{3}}\right)^{2},\left(\frac{t_{2}}{t_{3}}\right)^{2}\right)
\end{aligned}
$$

Thus, $G=\left\{( \pm \lambda, \pm \lambda, \lambda): \lambda \in \mathbb{C}^{\times}\right\}$and it acts on $\mathbb{C}^{3}-0$ by multiplication. The quotient $\left[\mathbb{C}^{3}-0 / G\right]$ is a stacky $\mathbb{P}^{2}$.

## A. 5 Line bundles and divisors on toric stacks

A line bundle on the quotient stack $X_{N}=\left[U_{\Sigma} / G\right]$ is the same thing as a $G$-equivariant line bundle on $U_{\Sigma}$. The Picard group of $U_{\Sigma}$ is trivial, so line bundles on $X_{N}$ correspond to the various $G$ linearizations of $\mathcal{O}_{U_{\Sigma}}$.
Proposition A. 2 (Borisov and Hua, 2009 BH09, Proposition 3.3]). There is an isomorphism, describing the Picard group of $X_{N}$ via divisors $D_{\rho}$ :

$$
\begin{aligned}
\mathbb{Z}^{\Sigma(1)} / \beta^{*} \mathrm{M} & \cong \operatorname{Pic} X_{N} \\
\left(b_{\rho}\right)_{\rho} & \mapsto \mathcal{O}_{X_{N}}\left(\sum_{\rho \in \Sigma(1)} \frac{b_{\rho}}{\left|E_{\rho}\right|} D_{\rho}\right)
\end{aligned}
$$

The line bundle $\mathcal{O}_{X_{N}}\left(\sum_{\rho \in \Sigma(1)} \frac{b_{\rho}}{\left|E_{\rho}\right|} D_{\rho}\right)$ is the trivial line bundle $\mathcal{O}_{U_{\Sigma}}=U_{\Sigma} \times \mathbb{C}$ with the $G$ linearization

$$
\begin{aligned}
G \times\left(U_{\Sigma} \times \mathbb{C}\right) & \rightarrow U_{\Sigma} \times \mathbb{C} \\
\lambda \cdot(z, t) & \mapsto\left(\lambda \cdot z, t \prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}}\right) .
\end{aligned}
$$

Let $D=\sum_{\rho \in \Sigma(1)} \frac{b_{\rho}}{\left|E_{\rho}\right|} D_{\rho}$ be a divisor at infinity on $X_{N}$. We assign to $D$ a polygon $P_{D}$ in $\mathrm{M}_{\mathbb{R}}$ defined by the intersection of the half planes provided by the coefficients of $D$ :

$$
\begin{equation*}
P_{D}:=\bigcap_{\rho \in \Sigma(1)}\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle \geq-\frac{b_{\rho}}{\left|E_{\rho}\right|}\right\} \tag{52}
\end{equation*}
$$

A global section of a line bundle on $\mathcal{X}_{N}$ is the same thing as a $G$-invariant global section of $\mathcal{O}_{U_{\Sigma}}$. As in the case of toric varieties, global sections of toric line bundles are identified with integral points in the associated polygons:

Proposition A. 3 (Borisov and Hua, 2009 [BH09, Proposition 4.1]). We have

$$
H^{0}\left(X_{N}, \mathcal{O}_{x_{N}}(D)\right) \cong \bigoplus_{m \in P_{D} \cap \mathrm{M}} \mathbb{C} \cdot \chi^{m}
$$

The $G$-invariant section of $\mathcal{O}_{U_{\Sigma}}$ corresponding to $\chi^{m}, m \in P_{D} \cap \mathrm{M}$, is $\prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}}$, where $a_{\rho}=$ $\left|E_{\rho}\right|\left\langle m, u_{\rho}\right\rangle+b_{\rho}$.

Proof. We have $H^{0}\left(U_{\Sigma}, \mathcal{O}_{U_{\Sigma}}\right)=\mathbb{C}\left[z_{\rho}: \rho \in \Sigma(1)\right]$. The global section $\prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}}$ is $G$-invariant if and only if

$$
\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}} \cdot \prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}}=\prod_{\rho \in \Sigma(1)}\left(z_{\rho} \lambda_{\rho}\right)^{a_{\rho}} \text { for all } \rho \in \Sigma(1)
$$

which is equivalent to the equations $\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}-a_{\rho}}=1$ for all $\rho \in \Sigma(1)$. By exactness of (50), this is equivalent to the existence of $m \in \mathrm{M}$ such that $a_{\rho}-b_{\rho}=\left|E_{\rho}\right|\left\langle m, u_{\rho}\right\rangle$ for all $\rho \in \Sigma(1)$.

## B Combinatorial rules for the linear system of equations $\mathbb{V}_{b w}$

In this appendix, we collect some combinatorial rules that facilitate the computation of the small polygons and equations in $\mathbb{V}_{\text {bw }}$.

## B. 1 Equivalent description of the small polygons

Consider the lines

$$
\begin{equation*}
L_{\rho}:=\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle=-b_{\rho}\right\} \tag{53}
\end{equation*}
$$

that form the boundary of the small Newton polygon $N_{\mathrm{bw}}$. We give an alternate description of these lines. Recall that $\widetilde{\Gamma}$ be the biperiodic graph on the plane given by the lift of $\Gamma$ to the universal cover of $\mathbb{T}$. The zig-zag paths in $\widetilde{\Gamma}$ for a given $\rho$ divide the plane into an infinite collection of strips $\mathcal{S}_{\rho}(d)$ parameterized by $d \in \frac{1}{\left|E_{\rho}\right|} \mathbb{Z}$ such that

$$
\mathcal{S}_{\rho}(d) \cap V(\widetilde{\Gamma})=\left\{\mathrm{v} \in V(\widetilde{\Gamma}):\left[D_{\rho}\right] \mathbf{D}(\mathrm{v})=d\right\}
$$

where for a divisor $D,\left[D_{\rho}\right] D$ denotes the coefficient of $D_{\rho}$ in $D$. We assign to each strip $\mathcal{S}_{\rho}(d)$ a line $L_{\rho}(d)$ in $\mathrm{M}_{\mathbb{R}}$ parallel to $E_{\rho}$, using the following rule illustrated on Figure 13 ,

1. The line associated to a strip $\mathcal{S}_{\rho}(d)$ contains the side $E_{\rho}$ if and only if either
i) The strip $\mathcal{S}_{\rho}(d)$ is on the right (when facing in the direction of the path) of a zig-zag path $\alpha_{1} \in Z_{\rho}$, and $\alpha_{1}$ contains b.
ii) The strip $\mathcal{S}_{\rho}(d)$ contains b , and b is not in a zig-zag path in $Z_{\rho}$, or
2. Moving to the strip to the left shifts the line $1 /\left|E_{\rho}\right|$ steps to the left.

We call the strip to the left of the one whose line contains $E_{\rho}$, and all strips obtained by its translations by $H_{1}(\mathbb{T}, \mathbb{Z})$, exceptional strips.

Proposition B.1. If we associate lines to strips as above, the boundary of the small Newton polygon $N_{\mathrm{bw}}$ is given by the lines $\left\{L_{\rho}\left(d_{\rho}\right)\right\}$, where $d_{\rho} \in \frac{1}{\left|E_{\rho}\right|} \mathbb{Z}$ is determined by the condition $\mathrm{w} \in \mathcal{S}_{\rho}\left(d_{\rho}\right)$, that is, it is the index of the strip containing w in the direction $\rho$.

Proof. In order for the line $L_{\rho}$ in (53) to contain $E_{\rho}$, we must have $b_{\rho}=0$, where $b_{\rho}$ is the coefficient of $D_{\rho}$ in (20). We have to consider two cases.

1. There is a zig-zag path $\alpha \in Z_{\rho}$ such that b is contained in $\alpha$. We need $\left[D_{\rho}\right](\mathbf{D}(\mathrm{b}))=$ $\frac{1}{\left|E_{\rho}\right|}+\left[D_{\rho}\right](\mathbf{D}(\mathrm{w}))$, which means w is contained in the strip $\mathcal{S}$ to the right of the one containing b , with $\alpha$ separating the two strips.


Figure 13: The lifts of zig-zag paths $\alpha_{1}, \ldots, \alpha_{k}$ in $Z_{\rho}$ divide the plane into strips. The side $L_{\rho}$ of the small polygon $N_{\text {bw }}$ and the columns of the matrix $\mathbb{V}_{\text {bw }}$ are determined by the strips containing b and w . The black vertex b is the black dot. On the left panel, b is on a zig-zag path, and on the right, it is between two zig-zag paths. Written inside each strip in blue is the subset of $Z_{\rho}$ that gives rise to equations in $\mathbb{V}_{\mathrm{bw}}$ if w is contained in that strip. Exceptional strips are shaded.
2. No zig-zag path in $Z_{\rho}$ contains b. In this case, we need the coefficients $\left[D_{\rho}\right](\mathbf{D}(\mathrm{b}))=$ $\left[D_{\rho}\right](D(\mathrm{w}))$, which means w is in the strip $\mathcal{S}$ containing b .

If $\mathrm{w}_{2}$ is a white vertex in the strip to the left of the strip containing a white $\mathrm{w}_{1}$, then $\left[D_{\rho}\right]\left(\mathbf{D}\left(\mathrm{w}_{2}\right)\right)=$ $\left[D_{\rho}\right]\left(\mathbf{D}\left(\mathrm{w}_{1}\right)\right)+\frac{1}{\left|E_{\rho}\right|}$. So if we define $b_{\rho}\left(\mathrm{w}_{1}\right)$ and $b_{\rho}\left(\mathrm{w}_{2}\right)$ as in (20) with $\mathrm{w}=\mathrm{w}_{1}$ and $\mathrm{w}=\mathrm{w}_{2}$ respectively, then $b_{\rho}\left(\mathrm{w}_{2}\right)=b_{\rho}\left(\mathrm{w}_{1}\right)+\frac{1}{\mid E_{\rho}}$. Note that the line (53) which bounds $N_{\mathrm{bw}_{2}}$ is given by

$$
L_{\rho}\left(\mathrm{w}_{2}\right):=\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle=b_{\rho}\left(\mathrm{w}_{2}\right)\right\} .
$$

The similar line which bounds $N_{\mathrm{bw}_{1}}$ is

$$
L_{\rho}\left(\mathrm{w}_{1}\right):=\left\{m \in \mathrm{M}_{\mathbb{R}}:\left\langle m, u_{\rho}\right\rangle=b_{\rho}\left(\mathrm{w}_{1}\right)\right\},
$$

so the line $L_{\rho}\left(\mathrm{w}_{2}\right)$ is obtained from the line $L_{\rho}\left(\mathrm{w}_{1}\right)$ by shifting $1 /\left|E_{\rho}\right|$ steps to the left.

## B. 2 The equations in $\mathbb{V}_{\text {bw }}$

We describe the equations of type 2 in Section 3.1.2
Let $\rho \in \Sigma(1)$ be a ray and let $Z_{\rho}=\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}$, where $\alpha_{1}, \ldots, \alpha_{k}$ are labeled in cyclic order. Their lifts to the universal cover of the torus divides it into strips, see Figure 13 . We denote by $\mathcal{S}_{i}$ the strip immediately to the right of $\alpha_{i}$.

Proposition B.2. The set of extra linear equations is described as follows:

1. One of these zig-zag paths contains b . We can assume it is $\alpha_{1}$. Then the subset of $Z_{\rho}$ that contributes an equation to $\mathbb{V}_{\mathrm{bw}}$ is:

$$
\begin{align*}
& \text { empty if } \mathrm{w} \in \mathcal{S}_{k} \text {; } \\
& \alpha_{i+1}, \ldots, \alpha_{k} \text { if } \mathrm{w} \in \mathcal{S}_{i} \text { for some } i \neq k \text {. } \tag{54}
\end{align*}
$$

2. The vertex b is not in any of zig-zag paths in $Z_{\rho}$. Then the subset of $Z_{\rho}$ is

$$
\begin{align*}
\alpha_{1}, \ldots, \alpha_{k} & \text { if } \mathrm{w} \in \mathcal{S}_{k} \\
\alpha_{i+1}, \ldots \alpha_{k} & \text { if } \mathrm{w} \in \mathcal{S}_{i}, \text { for some } i \neq k \tag{55}
\end{align*}
$$

Proof. Plugging

$$
Y_{\mathrm{bw}}=D_{N}-\mathbf{D}(\mathrm{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}
$$

into (44), we first observe that (44) does not change if we replace w by its any translate on the universal cover because $\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b})$ changes by the same amount as $\left.\left[E_{\mathrm{bw}}\right]\right|_{\mathcal{C}}$ but with the opposite sign. Therefore we may assume that among all its possible translates in the universal cover, the strip $\mathcal{S}_{i}$ is the one that is immediately to the right of b . Then we have

$$
\left.(\mathbf{d}(\mathrm{w})-\mathbf{d}(\mathrm{b}))\right|_{\mathcal{C} \cap D_{\rho}}=-\nu\left(\alpha_{1}\right)-\cdots-\nu\left(\alpha_{i}\right)
$$

and the coefficient of $D_{\rho}$ in $(\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b}))$ is $-\frac{i}{k}$.
Now we distinguish two cases:

1. Suppose b is contained in $\alpha_{1}$, so that the coefficient of $D_{\rho}$ in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}$ is $\frac{1}{k}$. Then we have

$$
\left.\left[-\mathbf{D}(\mathrm{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right\rfloor\right|_{\mathcal{C} \cap D_{\rho}}=0
$$

so that (44) is $\nu\left(\alpha_{i+1}\right)+\cdots+\nu\left(\alpha_{k}\right)$, which proves (54).
2. Suppose b is not contained in any of the $\alpha_{i}$, so that the coefficient of $D_{\rho}$ in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}$ is 0 . We have

$$
\left.\left\lfloor-\mathbf{D}(\mathrm{w})+\mathbf{D}(\mathrm{b})-\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{\left|E_{\rho}\right|} D_{\rho}\right\rfloor\right|_{\mathcal{C} \cap D_{\rho}}= \begin{cases}0 & \text { if } i \neq k, \\ \sum_{j=1}^{k} \nu\left(\alpha_{j}\right) & \text { if } i=k .\end{cases}
$$

This gives

$$
(44)= \begin{cases}\nu\left(\alpha_{i+1}\right)+\cdots+\nu\left(\alpha_{k}\right) & \text { if } i \neq k \\ \sum_{j=1}^{k} \nu\left(\alpha_{j}\right) & \text { if } i=k\end{cases}
$$

We obtain (55).

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