The inverse spectral map for dimers

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August 25, 2023

Abstract

In 2015, Vladimir Fock proved that the spectral transform, associating to an element of a dimer cluster integrable system its spectral data, is birational by constructing an inverse map using theta functions on Jacobians of spectral curves. We provide an alternate construction of the inverse map that involves only rational functions in the spectral data.

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1 Introduction

The planar dimer model is a classical statistical mechanics model, involving the study of the set of *dimer covers* (perfect matchings) of a planar, edge-weighted graph. In the 1960s, Kasteleyn [Kas61, Kas63] and Temperley and Fisher [TF61] showed how to compute the (weighted) number of dimer covers of planar graphs using the determinant of a signed adjacency matrix now known as the *Kasteleyn matrix*.

In mathematics the dimer model was popularized with the papers [EKLP92a, EKLP92b] on the "Aztec diamond" and later with results on the local statistics [Ken97], conformal invariance [Ken00], and limit shapes [CKP01], connections with algebraic geometry [KOS06, KO06], cluster varieties and integrability [GK13], and string theory [HK05].

While the dimer model can be considered from a purely combinatorial point of view, it also has a rich integrable structure, first described in [GK13]. The integrable structure on dimers on graphs on the torus was found to generalize many well-known integrable systems, see for example [FM16] and [AGR21]. What is especially important is that the related integrable system is of cluster nature, and this allows one to immediately quantize it, getting a quantum integrable system.

From the point of view of classical mechanics, associated to the dimer model on a bipartite graph on a torus (or equivalently a periodic bipartite planar graph) is a Poisson variety with a Hamiltonian integrable system. Underlying this system is an algebraic curve $C = \{P(z, w) = 0\}$ (called the *spectral curve*) and a divisor on this curve–essentially a set of g distinct points $\{(p_1, q_1), \ldots, (p_g, q_g)\}$ on C. This is the *spectral data* associated to the model. It was shown in [KO06] that the map from the weighted graph to the spectral data was bijective, from the space of "face weights" (see below) to the moduli space of genus-g curves and effective degree-g divisors on the open spectral curve C° . Subsequently Fock [Foc15] constructed the inverse spectral map (from the spectral data to the face weights), describing it in terms of theta functions over the spectral curve. The special case of genus 0 was described earlier in [Ken02, KO06] and an explicit construction in the case of genus 1 was more recently given in [BCdT20]. Positivity of Fock's inverse map was studied in [BCdT21]. In the current paper, we show that the inverse map can be given an explicit rational expression in terms of the divisor points $(p_i, q_i) \in C^\circ$ and the points of C at toric infinity. An exact statement is given in Theorem 3.10 below.

While Fock's construction is very natural and interacts nicely with positivity, it involves theta functions. Our construction gives the inverse map as ratios of certain determinants in the spectral data and can be explicitly computed using computer algebra. We briefly describe our construction now. The spectral data is defined via a matrix K = K(z, w) called the Kasteleyn matrix, whose rows are indexed by white vertices, columns by black vertices, and whose entries are Laurent polynomials in z and w. Let us consider the adjugate matrix of K:

$$Q = Q(z, w) = K^{-1} \det K.$$

The matrix Q is important when studying the probabilistic aspects of the dimer model (on the lift of the graph on the torus to the plane): the edge occupation variables form a determinantal process whose kernel is given by the Fourier coefficients of Q/P, as discussed in [KOS06]. In the present work, we have a different use for Q: finding (a column of) the matrix Q from the spectral data allows us to reconstruct the face weights and thereby invert the spectral transform.

The points $(p_i, q_i) \in \mathcal{C}$ are defined to be the points where a column of Q, corresponding to a fixed white vertex \mathbf{w} , vanishes. We show that entries in the \mathbf{w} -column of Q, which are Laurent polynomials, can be reconstructed from the spectral data by solving a linear system of equations. Some of the linear equations are easy to describe: for any black vertex b, we have $Q_{\mathbf{bw}}(p_i, q_i) = 0$ for $i = 1, \ldots, g$, which are g linear equations in the coefficients of the Laurent polynomial $Q_{\mathbf{bw}}$. However, these equations are usually not sufficient to determine the coefficients of $Q_{\mathbf{bw}}$. We find additional equations from the vanishing of $Q_{\mathbf{bw}}$ at certain points at infinity of the spectral curve \mathcal{C} , and show that these equations determine $Q_{\mathbf{bw}}$ uniquely, up to a non-zero constant. We then give a procedure to reconstruct the weights from the \mathbf{w} -column of Q.

A key construction in our approach is the extension of the Kasteleyn matrix K to a map of vector bundles on a toric stack, for which we make crucial use of the classification of line bundles on toric stacks and the computation of their cohomology developed in [BH09]. Toric stacks already appear implicitly in the context of the spectral transform in [KO06] and explicitly [TWZ19].

The article is organized as follows. In Section 2 we review the dimer cluster integrable system and the spectral transform. In Section 3, we state Theorem 3.1, which is our main result, and describe the reconstruction procedure. We work out two detailed examples in Section 4. Sections 5, 6 and 7 contain proofs of our results. In Appendix A, we review results from toric geometry. In Appendix B, we provide explicit combinatorial descriptions for some of our constructions. These are useful for computations.

Acknowledgments. The work of A.G. was supported by the NSF grants DMS-1900743, DMS-2153059. Work of R. K. was supported by NSF grant DMS-1940932 and the Simons Foundation grant 327929. We thank the referee for their careful reading of the manuscript and their numerous suggestions.

2 Background

For further information about the material in this section see [GK13].

2.1 Dimer models

Let Γ be a bipartite graph on the torus $\mathbb{T} \cong S^1 \times S^1$ such that the connected components of the complement of Γ —the faces—are contractible. We denote by $B(\Gamma)$ and $W(\Gamma)$ the black and white vertices of Γ , by $V(\Gamma)$ the vertices, and by $E(\Gamma)$ the edges of Γ . When the graph is clear from context, we will usually abbreviate these to B, W, V and E.

A dimer model on the torus is a pair $(\Gamma, [wt])$, where Γ is a bipartite graph on the torus as above and $[wt] \in H^1(\Gamma, \mathbb{C}^{\times})$ (Here and throughout the paper, \mathbb{C}^{\times} denotes the group of nonzero complex numbers under multiplication). For a loop L and a cohomology class [wt], we denote by [wt]([L])the pairing between the cohomology and the homology. We orient edges from their black vertex to their white vertex. The cohomology class [wt] can be represented by a cocycle wt which, using this orientation, can be identified with a \mathbb{C}^{\times} -valued function on the edges of Γ called an *edge weight*.

The edge weight is well-defined modulo multiplication by coboundaries, which are functions on edges e = bw given by $f(w)f(b)^{-1}$ for functions $f : V(\Gamma) \to \mathbb{C}^{\times}$. Note that the weight of a loop is not the product of its edge weights, but the "alternating product" of its edge weights: edges oriented against the orientation of the loop are multiplied with exponent -1.

A dimer cover or perfect matching m of Γ is a subset of $E(\Gamma)$ such that each vertex of Γ is incident to exactly one edge in m. Let \mathcal{M} denote the set of dimer covers of Γ . If we fix a reference dimer cover m₀, we get a function

$$\pi_{\mathbf{m}_0} : \mathcal{M} \to H_1(\mathbb{T}, \mathbb{Z})$$
$$\mathbf{m} \mapsto [\mathbf{m} - \mathbf{m}_0].$$

Here $m - m_0$ is the 1-chain which assigns 1 to (oriented) edges of m and -1 to (oriented) edges of m_0 , so $m - m_0$ is a union of oriented cycles and doubled edges, whose homology class is $[m - m_0]$.

The Newton polygon of Γ is the polygon

$$N(\Gamma) := \text{Convex-hull}(\pi_{\mathrm{m}_0}(\mathcal{M})) \subset H_1(\mathbb{T}, \mathbb{R})$$

defined modulo translation by $H_1(\mathbb{T},\mathbb{Z})$. Changing the reference dimer cover from m_0 to m'_0 results in a translation of the polygon by $[m_0 - m'_0]$, so the Newton polygon does not depend on the choice.

We assume that Γ is such that $N(\Gamma)$ has interior. This is a nondegeneracy condition on Γ . (When N has empty interior, the graph Γ is equivalent under certain elementary transformations to a graph whose lift to \mathbb{R}^2 is disconnected, that is, has noncontractible faces; such a graph breaks into essentially one-dimensional components, and there is no integrable system.)

2.2 Zig-zag paths and the Newton polygon

A zig-zag path in Γ is a closed path that turns maximally right at each black vertex and maximally left at each white vertex. The medial graph of Γ is the graph Γ^{\times} that has a vertex v_e at the midpoint of each edge e of Γ and an edge between v_e and $v_{e'}$ whenever e and e' occur consecutively around a face of Γ . Note that by construction, each vertex of Γ^{\times} has degree 4. A zig-zag path in Γ corresponds to a cycle in Γ^{\times} that goes straight through each degree four vertex, i.e., at every vertex, the outgoing edge of the cycle is the one that is opposite the incoming one (see Figure 2). Hereafter, when we say zig-zag path, we mean the corresponding cycle in the medial graph.

Let Γ be the biperiodic graph on the plane given by the lift of Γ to the universal cover of \mathbb{T} . The bipartite graph Γ is said to be *minimal* if the lift of any zig-zag path does not self-intersect, and lifts of any two zig-zag paths do not have "parallel bigons", where by *parallel bigon* we mean two



Figure 1: The fundamental rectangle R, along with the cycles γ_z, γ_w .



Figure 2: A zig-zag path in a graph Γ and the corresponding cycle in the medial graph Γ^{\times} .

consecutive intersections where both paths are oriented in the same direction from one to the next. For a minimal bipartite graph Γ on the torus, the Newton polygon has an alternative description in terms of the zig-zag paths of Γ . Namely, since Γ is embedded in \mathbb{T} , each zig-zag path α has a non-zero homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z})$. The polygon $N(\Gamma)$ is the unique convex integral polygon defined modulo translation in $H_1(\mathbb{T}, \mathbb{Z})$ whose integral primitive edge vectors in counterclockwise order around N are given by the vectors $[\alpha]$ for all zig-zag paths α .

Example 2.1. Consider the fundamental domain for the square lattice shown in Figure 1, and let γ_z, γ_w be cycles generating $H_1(\mathbb{T}, \mathbb{Z})$ as shown there. We will write homology classes in $H_1(\mathbb{T}, \mathbb{Z})$ in the basis (γ_z, γ_w) . There are four zig-zag paths labeled α, β, γ and δ with homology classes (-1, 1), (-1, -1), (1, -1) and (1, 1) respectively (Figure 3), and therefore the Newton polygon is

Convex-hull{(1,0), (0,1), (-1,0), (0,-1)}.

2.3 The cluster variety assigned to a Newton polygon

For a convex integral polygon $N \subset H_1(\mathbb{T}, \mathbb{R})$ defined modulo translation, consider the family of minimal bipartite graphs Γ with Newton polygon $N(\Gamma) = N$. Any two graphs Γ_1, Γ_2 in the family are related by certain *elementary transformations*; see Figure 4. An elementary transformation $\Gamma_1 \to \Gamma_2$ gives rise to a birational map $H^1(\Gamma_1, \mathbb{C}^{\times}) \dashrightarrow H^1(\Gamma_2, \mathbb{C}^{\times})$. Gluing the tori $H^1(\Gamma, \mathbb{C}^{\times})$ by these maps, we obtain a space \mathcal{X}_N , called the *dimer cluster Poisson variety*. It carries a canonical



Figure 3: Zig-zag paths and Newton polygon for the bipartite graph in Figure 1.





 $contraction\ uncontraction\ move$

Figure 4: The elementary transformations.

Poisson structure. The Poisson center is generated by the loop weights of the zig-zag paths. The space \mathcal{X}_N is the phase space of the cluster integrable system. See details in [GK13].

2.4 Some notation

Let Σ denote the normal fan of N (see Section A.2 and Figures 7 and 10) so that the set of rays $\Sigma(1) = \{\rho\}$ of Σ is in bijection with the set of edges of N. We denote the edge of N whose inward normal is directed along the ray ρ by E_{ρ} , and the primitive vector along ρ by u_{ρ} .

Let $M := H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ and $M^{\vee} := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^2$ be dual lattices and let $\langle *, * \rangle : M \times M^{\vee} \to \mathbb{Z}$ denote the duality pairing. Let us consider the algebraic torus with lattice of characters M:

$$T := Hom_{\mathbb{Z}}(M, \mathbb{C}^{\times}) \cong (\mathbb{C}^{\times})^2$$

Let $M_{\mathbb{R}}$ (resp. $M_{\mathbb{R}}^{\vee}$) denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $M^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$), so that $N \subset M_{\mathbb{R}}$ and $\Sigma \subset M_{\mathbb{R}}^{\vee}$.

An elementary transformation $\Gamma_1 \to \Gamma_2$ induces a canonical bijection between zig-zag paths in Γ_1 and zig-zag paths in Γ_2 . Therefore, the set of zig-zag paths is canonically associated with N. We denote the set of zig-zag paths by Z, and for an edge E_{ρ} of N, we denote by Z_{ρ} the set of zig-zag paths α such that the primitive vector $[\alpha]$ is contained in E_{ρ} .

2.5 The Kasteleyn matrix

Let R be a fundamental rectangle for \mathbb{T} , so that \mathbb{T} is obtained by gluing together opposite sides of R. Let γ_z, γ_w be the oriented sides of R generating $H_1(\mathbb{T}, \mathbb{Z})$, as shown in Figure 1. Let z (resp. w) denote the character χ^{γ_w} (resp. χ^{γ_z}), so the coordinate ring of \mathbb{T} is $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$.

Let $(*, *)_{\mathbb{T}}$ be the intersection pairing on $H_1(\mathbb{T}, \mathbb{Z})$. For $z, w \in \mathbb{C}^{\times}$ we multiply edge weights on edges crossing γ_z by $z^{\pm 1}$ and those crossing γ_w by $w^{\pm 1}$, with the sign determined by the orientation. Precisely, we multiply by

$$\phi(e) := z^{(e,\gamma_w)_{\mathbb{T}}} w^{(e,-\gamma_z)_{\mathbb{T}}},\tag{1}$$

Here $(e, *)_{\mathbb{T}} := (l_e, *)_{\mathbb{T}}$ is the intersection index with the oriented loop l_e obtained by concatenating e = bw with an oriented path contained in R from w to b.

A Kasteleyn sign is a cohomology class $[\epsilon] \in H^1(\Gamma, \mathbb{C}^{\times})$ such that for any loop L in Γ , $[\epsilon]([L])$ is -1 (resp., 1) if the number of edges in L is 0 mod 4 (resp., 2 mod 4). Given edge weights wt and ϵ representing [wt] and $[\epsilon]$ respectively, one defines the Kasteleyn matrix K = K(z, w), whose columns and rows are parameterized by $b \in B$ and $w \in W$ respectively:

$$K_{\rm w,b} = \sum_{e \in E \text{ incident to b,w}} wt(e)\epsilon(e)\phi(e).$$
⁽²⁾

It describes a map of free $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ -modules, called the *Kasteleyn operator*:

$$K: \ \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^B \to \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^W,$$
(3)

$$\delta_{\mathbf{b}} \longmapsto \sum_{\mathbf{w} \in W} K_{\mathbf{w}, \mathbf{b}} \delta_{\mathbf{w}}.$$
 (4)

Theorem 2.2 (Kasteleyn 1963, [Kas63]). Fix a dimer cover m_0 , and let $\phi(m_0) = \prod_{e \in m_0} \phi(e)$. Then,

$$\frac{1}{wt(m_0)\epsilon(m_0)\phi(m_0)} \det K = \sum_{m \in \mathcal{M}} \operatorname{sign}([m - m_0])[wt]([m - m_0])\chi^{[m - m_0]},$$



Figure 5: Shown on the left is a labeling of vertices and faces of Γ , and two cycles a (red) and b (green) in Γ that generate $H_1(\mathbb{T},\mathbb{Z})$. Shown on the right is a cocycle representing [wt], along with ϵ and ϕ . The signs are due to ϵ , the z, w due to ϕ , and other weights are wt.

where $sign([m - m_0]) \in \{\pm 1\}$ is a sign that depends only on the homology class $[m - m_0]$ and $[\epsilon]$.

The characteristic polynomial is the Laurent polynomial

$$P(z,w) := \frac{1}{wt(\mathbf{m}_0)\epsilon(\mathbf{m}_0)\phi(\mathbf{m}_0)} \det K.$$

Its vanishing locus $C^{\circ} := \{P(z, w) = 0\} \subset (\mathbb{C}^{\times})^2$ is called the *(open part of the) spectral curve*. Theorem 2.2 implies that N is the Newton polygon of P(z, w). Although the definition of the Kasteleyn matrix uses cocycles representing the cohomology classes wt and ϵ , the spectral curve does not depend on these choices.

Example 2.3. Let a and b be the two cycles in Γ shown on the left of Figure 5 whose projections to \mathbb{T} generate $H_1(\mathbb{T}, \mathbb{Z})$. Let $[wt] \in H^1(\Gamma, \mathbb{C}^{\times})$ and let A := [wt]([a]), B := [wt]([b]). For i = 1, 2, 3, let X_i denote the $[wt]([\partial f_i])$, where ∂f_i denotes the boundary of the face f_i (the weight of the fourth face is determined by the fact that the product of all face weights is 1). Then (X_1, X_2, X_3, A, B) generate the coordinate ring of $H^1(\Gamma, \mathbb{C}^{\times})$. A cocycle representing [wt] is shown on the right of Figure 5, along with ϵ and ϕ . The Kasteleyn matrix and the spectral curve are:

$$K = \begin{pmatrix} b_1 & b_2 \\ 1 - Az & 1 - \frac{X_1 X_3}{B_W} \\ -1 + Bw & X_1 - \frac{B_1}{AX_2 z} \end{pmatrix} \stackrel{W_1}{W_2},$$
$$P(z, w) = \left(1 + X_1 + \frac{1}{X_2} + X_1 X_3\right) - Bw - \frac{X_1 X_3}{Bw} - \frac{1}{AX_2 z} - AX_1 z.$$
(5)

2.6 The toric surface assigned to a Newton polygon

In this section, we collect some notation regarding toric varieties, and refer the reader to the Appendices A.1 and A.2 for more details. A convex integral polygon $N \subset M_{\mathbb{R}}$ determines a

compactification X_N of the complex torus T called a *toric surface*, and a divisor D_N supported on the boundary $X_N - T$, so that Laurent polynomials with Newton polygon N extend naturally to sections of the coherent sheaf $\mathcal{O}_{X_N}(D_N)$ (for background on the coherent sheaf associated to a divisor, see for example [CLS11, Chapter 4]).

Denote by $|D_N|$ the projective space of non-zero global sections of the coherent sheaf $\mathcal{O}_{X_N}(D_N)$, considered modulo a multiplicative constant. Assigning to a section its vanishing locus, we see elements of $|D_N|$ as curves in X_N whose restrictions to T are defined by Laurent polynomials with Newton polygon contained in N.

The genus g of the generic curve in $|D_N|$ is equal to the number of interior lattice points in N. Recall that the edges $\{E_{\rho}\}$ of N are in bijection with the rays $\{\rho\}$ of Σ . Each edge E_{ρ} of N determines a projective line D_{ρ} which we call a *line at infinity* of X_N , and

$$X_N - \mathcal{T} = \bigcup_{\rho \in \Sigma(1)} D_{\rho}.$$

The divisor D_N is given by

$$D_N = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,\tag{6}$$

where $a_{\rho} \in \mathbb{Z}$ are such that

$$N = \bigcap_{\rho \in \Sigma(1)} \{ m \in \mathcal{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle \ge -a_{\rho} \}.$$
(7)

The lines D_{ρ} intersect according to the combinatorics of N: precisely, for $\rho_1, \rho_2 \in \Sigma(1)$, the intersection $D_{\rho_1} \cap D_{\rho_2}$ is empty if $E_{\rho_1} \cap E_{\rho_2}$ is empty and a point if $E_{\rho_1} \cap E_{\rho_2}$ is a vertex of N. The intersection index of a generic curve in $|D_N|$ with the line D_{ρ} is equal to the number $|E_{\rho}|$ of primitive integral vectors in the edge E_{ρ} . The points of intersection are called *points at infinity*. Let $\mathcal{C} \in |D_N|$ denote the compactification of the open spectral curve \mathcal{C}° , i.e., \mathcal{C} is the closure of \mathcal{C}° in X_N . \mathcal{C} is called the *spectral curve*.

2.7 Casimirs

Let α be a zig-zag path $\alpha = b_1 \rightarrow w_1 \rightarrow b_2 \rightarrow \cdots \rightarrow w_d \rightarrow b_1$ in Z_{ρ} . We define the *Casimir* C_{α} by

$$C_{\alpha} := (-1)^d [\epsilon]([\alpha])[wt]([\alpha])$$

The Casimirs determine points at infinity of C as follows: since $[\alpha]$ is primitive and $\langle u_{\rho}, [\alpha] \rangle = 0$, we can extend it to a basis (x_1, x_2) of M with $[\alpha] = x_1$ and $\langle x_2, u_{\rho} \rangle = 1$. The affine open variety in X_N corresponding to the cone ρ is

$$U_{\rho} = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, x_2] \cong \mathbb{C}^{\times} \times \mathbb{C},$$

and $D_{\rho} \cap U_{\rho}$ is defined by $x_2 = 0$, and so the character $x_1^{-1} = \chi^{-[\alpha]}$ is a coordinate on the dense open torus $\mathbb{C}^{\times} = D_{\rho} \cap U_{\rho}$ in D_{ρ} . Therefore, the equation

$$\chi^{-[\alpha]}(\nu_{\rho}(\alpha)) = C_{\alpha},\tag{8}$$

defines a point $\nu_{\rho}(\alpha)$ in D_{ρ} . In other words, the point is defined as the unique point on the line at infinity such that the monomial $z^i w^j$, where $-[\alpha] = (i, j)$, evaluates to C_{α} . We will prove later (see (41)) that these are precisely the points at infinity of C. **Example 2.4.** Consider the fundamental domain of the square lattice, whose zig-zag paths were listed in Example 2.1 and Figure 3. The Casimirs are

$$C_{\alpha} = -\frac{B}{AX_1}, \quad C_{\beta} = -\frac{1}{ABX_2}, \quad C_{\gamma} = -\frac{AX_1X_2X_3}{B}, \quad C_{\delta} = -\frac{AB}{X_3}.$$
 (9)

Let us denote the normal ray in Σ of a zig-zag path ω by $\rho(\omega)$, so $u_{\rho(\alpha)} = (-1, -1)$ etc. We choose $x_2 = \chi^{(0,-1)}$ so that $\langle (0,-1), u_{\rho(\alpha)} \rangle = 1$. Then we have $U_{\rho(\alpha)} = \text{Spec} \mathbb{C}[x_1 = z^{-1}w, x_2 = w^{-1}]$ and $D_{\rho(\alpha)} \subset U_{\rho(\alpha)}$ is given by $x_2 = 0$. In this case, $D_N = D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}$ and P(z,w) is a global section of $\mathcal{O}_{X_N}(D_N)$. We trivialize $\mathcal{O}_{X_N}(D_N)$ over $U_{\rho(\alpha)}$ as follows:

$$\mathcal{O}_{X_N}(D_N)\big|_{U_{\rho(\alpha)}} = \{t \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}] : \text{div } t\big|_{U_{\rho(\alpha)}} + D_{\rho(\alpha)} \ge 0\} \cong \mathcal{O}_{U_{\rho(\alpha)}}$$
$$t \mapsto tx_2$$

Then making the change of variables $z = \frac{1}{x_1 x_2}$ and $w = \frac{1}{x_2}$, and multiplying by x_2 , the portion of the spectral curve C in U_{ρ} is cut out by

$$\left(1 + X_1 + \frac{1}{X_2} + X_1 X_3\right) x_2 - B - \frac{X_1 X_3}{B} x_2^2 - \frac{x_1 x_2^2}{A X_2} - \frac{A X_1}{x_1},$$

so that $\mathcal{C} \cap D_{\rho(\alpha)}$ is given by

$$-B - \frac{AX_1}{x_1} = 0.$$

Therefore, $\nu(\alpha)$ is given by $\frac{z}{w} = \frac{1}{x_1} = C_{\alpha}$, which agrees with (8). The table below lists the points at infinity for each of the zig-zag paths.

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity	
α	(-1, 1)	(-1, 1), (0, -1)	$x_1 = \frac{1}{C_\alpha}, x_2 = 0$	
β	(-1, -1)	(-1, -1), (0, -1)	$x_1 = \frac{1}{C_\beta}, x_2 = 0$	(10)
γ	(1, -1)	(1, -1), (0, 1)	$x_1 = \frac{1}{C_{\gamma}}, x_2 = 0$	
δ	(1, 1)	(1,1), (0,1)	$x_1 = \frac{1}{C_\delta}, x_2 = 0$	

2.8 The spectral transform

Our next goal is to define the spectral transform, which plays the key role in this paper. We present two equivalent definitions of the spectral transform. The first is the original definition of Kenyon and Okounkov [KO06], and it is the one which we use in computations. However, it depends on the choice of the distinguished white vertex \mathbf{w} . The second is more invariant, and does not require choosing a distinguished white vertex \mathbf{w} .

Recall that for each edge E_{ρ} of N, we have $\#Z_{\rho} = \#\mathcal{C} \cap D_{\rho}$, but there is no canonical bijection between these sets. We define a *parameterization of the points at infinity by zig-zag paths* to be a choice of bijections $\nu = \{\nu_{\rho}\}_{\rho \in \Sigma(1)}$, where

$$\nu_{\rho}: Z_{\rho} \xrightarrow{\sim} \mathcal{C} \cap D_{\rho}. \tag{11}$$

For a curve $C \in |D_N|$, we denote by $\text{Div}_{\infty}(C)$ the abelian group of divisors on C supported at the *infinity*, that is at $C \cap D_N$.

Compactifications of the Kasteleyn operator will play a important role in this paper. The main ingredient in the construction of these compactifications is a combinatorial object called the *discrete* Abel map introduced by Fock [Foc15] that encodes intersections with zig-zag paths. Let Γ be a minimal bipartite graph in \mathbb{T} with Newton polygon N and spectral curve \mathcal{C} . The discrete Abel map

$$\mathbf{d}: B \cup W \cup F \to \operatorname{Div}_{\infty}(\mathcal{C})$$

assigns to each vertex and face of Γ a divisor at infinity. It is defined uniquely up to a constant by the requirement that for a path γ from x to y, contained in the fundamental domain R, where x and y are either vertices or faces of Γ , we have

$$\mathbf{d}(y) - \mathbf{d}(x) = \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}} (\alpha, \gamma)_{R} \nu_{\rho}(\alpha).$$

Here $(\alpha, \gamma)_R$ is the intersection index in R, i.e., the signed number of intersections of α with γ . Since we require γ to be contained in R, this is well-defined, independent of the choice of path γ . Locally, the rule is as follows:

- 1. If b is a black vertex incident to a face f, and b and f are separated by $\alpha \in Z_{\rho}$, then $\mathbf{d}(\mathbf{b}) = \mathbf{d}(f) + \nu_{\rho}(\alpha)$.
- 2. If w is a white vertex incident to a face f, and w and f are separated by $\alpha \in Z_{\rho}$, then $\mathbf{d}(\mathbf{w}) = \mathbf{d}(f) \nu_{\rho}(\alpha)$.

We normalize **d**, setting the value of **d** at certain face f_0 of Γ to be 0. Then for any black vertex b, face f, and white vertex w of $\widetilde{\Gamma}$ we have:

$$\deg \mathbf{d}(\mathbf{b}) = 1, \ \deg \mathbf{d}(f) = 0, \ \deg \mathbf{d}(\mathbf{w}) = -1.$$

$$(12)$$

Remark 2.5. Only differences of the form $\mathbf{d}(y) - \mathbf{d}(x)$ will appear in our constructions later, so the choice of normalization does not play a role.

Example 2.6. Let us compute the discrete Abel map **d** for the square lattice in Figure 5. We normalize $\mathbf{d}(f_1) = 0$. Then we have

$$\mathbf{d}(\mathbf{b}_1) = \nu_{\rho(\gamma)}(\gamma), \quad \mathbf{d}(\mathbf{b}_2) = \nu_{\rho(\alpha)}(\alpha), \quad \mathbf{d}(\mathbf{w}_1) = -\nu_{\rho(\beta)}(\beta), \quad \mathbf{d}(\mathbf{w}_2) = -\nu_{\rho(\delta)}(\delta),$$

where ν is shown in table (10).

Definition 1. A divisor spectral data related to a Newton polygon N is a triple (\mathcal{C}, S, ν) where $\mathcal{C} \in |D_N|$ is a genus g curve on the toric surface X_N , S is a degree g effective divisor in \mathcal{C}° , and $\nu = \{\nu_\rho\}$ are parameterizations of the divisors $D_\rho \cap \mathcal{C}$, see (11). Denote by S_N the moduli space parameterizing the divisor spectral data on N. Let us fix a distinguished white vertex \mathbf{w} of Γ . Then there is a rational map (here and in the sequel, $-\rightarrow$ means a rational map), called the spectral transform, defined by Kenyon and Okounkov [KO06],

$$\kappa_{\Gamma,\mathbf{w}}: \mathcal{X}_N \dashrightarrow \mathcal{S}_N \tag{13}$$

defined on the dense open subset $H^1(\Gamma, \mathbb{C}^{\times})$ of \mathcal{X}_N by $[wt] \mapsto (\mathcal{C}, S, \nu)$ as follows:

- 1. C is the spectral curve.
- 2. For generic [wt], C is a smooth curve and coker K is the pushforward of a line bundle on C° . Let $s_{\mathbf{w}}$ be the section of coker K given by the **w**-entry of the cokernel map. S is defined to be the divisor of this section. In Corollary 6.3, we show that S has degree g. Then S is the set of g points in C° where the **w**-column of the adjugate matrix $Q = Q(z, w) = K^{-1} \det K$ vanishes.
- 3. The parameterization of points at infinity by zig-zag paths ν is defined as follows: $\nu_{\rho}(\alpha)$ is the point in $\mathcal{C} \cap D_{\rho}$ satisfying $\chi^{-[\alpha]} = C_{\alpha}$ (see Section 2.7). We call $\nu_{\rho}(\alpha)$ the point at infinity associated to α .

Definition 2. A line bundle spectral data related to a Newton polygon N is a triple $(\mathcal{C}, \mathcal{L}, \nu)$ where $\mathcal{C} \in |D_N|$ is a genus g curve on the toric surface X_N , \mathcal{L} is a degree g - 1 line bundle on \mathcal{C} , and ν is a parameterization of points at infinity by zig-zag paths. Denote by \mathcal{S}'_N the moduli space parameterizing the line bundle spectral data on N.

The spectral transform is a rational map

$$\kappa_{\Gamma,\mathbf{d}}:\mathcal{X}_N \dashrightarrow \mathcal{S}'_N$$

defined on the dense open subset $H^1(\Gamma, \mathbb{C}^{\times})$ of \mathcal{X}_N by $[wt] \mapsto (\mathcal{C}, \mathcal{L}, \nu)$, where:

- 1. C is the spectral curve.
- 2. Let $K|_{\mathcal{C}^{\circ}}$ denote the restriction of the Kasteleyn matrix to \mathcal{C}° . The discrete Abel map **d** determines an extension \overline{K} of $K|_{\mathcal{C}^{\circ}}$ to a morphism of locally free sheaves on \mathcal{C} ; see Section 6. The coherent sheaf \mathcal{L} is defined as the cokernel of \overline{K} . When \mathcal{C} is a smooth curve, which happens for generic [wt], \mathcal{L} is a line bundle. The convention deg $\mathbf{d}(\mathbf{w}) = -1$ implies that deg $\mathcal{L} = g 1$; see Proposition 6.4.
- 3. The parameterizations of the divisors $D_N \cap \mathcal{C}$ are defined by associating to a zig-zag path α the point at infinity $\nu_{\rho}(\alpha)$.

Since ρ is determined by α , we will use the simpler notation $\nu(\alpha) := \nu_{\rho}(\alpha)$ hereafter.

The two types of spectral data are equivalent. Given a degree g effective divisor S, we have (Proposition 6.4)

$$\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}\left(S + \mathbf{d}(\mathbf{w})\right). \tag{14}$$

On the other hand, given a line bundle \mathcal{L} and a white vertex \mathbf{w} , we can recover S as follows. Consider the Abel-Jacobi map

$$A^{g}: \operatorname{Sym}^{g} \mathcal{C} \to \operatorname{Jac}(\mathcal{C}),$$
$$E \mapsto \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(E + \mathbf{d}(\mathbf{w})).$$

Then A^g is birational by the Abel-Jacobi theorem [Bea13, Corollary 4.6]. We obtain $S = (A^g)^{-1}(\mathcal{O}_{\mathcal{C}})$.

Example 2.7. We compute the spectral transform for our running example of the square lattice. Let us take the distinguished white vertex to be $\mathbf{w} = w_1$.

$$Q = \begin{pmatrix} W_1 & W_2 \\ X_1 - \frac{1}{AX_2z} & -1 + \frac{X_1X_3}{Bw} \\ 1 - Bw & 1 - Az \end{pmatrix} \begin{array}{c} \mathbf{b}_1 \\ \mathbf{b}_2 \end{array}$$
(15)

Solving $Q_{b_1w}(p,q) = Q_{b_2w}(p,q) = 0$, we get

$$p = \frac{1}{AX_1X_2}, \quad q = \frac{1}{B}.$$
 (16)

Therefore, the spectral transform is:

$$\kappa_{\Gamma,\mathbf{w}}: H^1(\Gamma, \mathbb{C}^{\times}) \dashrightarrow \mathcal{S}_N$$
$$(X_1, X_2, X_3, A, B) \mapsto (\mathcal{C}, (p, q), \nu)$$

where $C = \{P(z, w) = 0\}$ with P(z, w) as is in (5), S = (p, q) is a single point (the genus g = 1 since N in Figure 3 has one interior lattice point) and ν is as shown in table (10).

3 The main theorem

Below we introduce functions V_{bw} on the moduli space S_N of spectral data, relying on results in the remaining Sections 5, 6, 7. They are defined for any pair $(b, w) \in B \times W$ of black and white vertices, and defined as the solution to a system of linear equations \mathbb{V}_{bw} .

The main result of the paper is the following.

Theorem 3.1. For the distinguished white vertex \mathbf{w} , the pull-back of the function V_{bw} under the spectral map coincides, up to a multiplicative constant, with the b \mathbf{w} matrix element Q_{bw} of the adjugate matrix $Q := K^{-1} \det K$ of the Kasteleyn matrix K. That is,

$$Q_{\mathbf{b}\mathbf{w}} = c \cdot \kappa^*_{\Gamma,\mathbf{w}}(\mathbf{V}_{\mathbf{b}\mathbf{w}}),\tag{17}$$

where c depends on b (and \mathbf{w}).

As an application of this result, we get an explicit description of the inverse to the spectral map (13); see Section 3.2.

The next few sections discuss the structure of the system of linear equations \mathbb{V}_{bw} . Detailed examples are given in Section 4.

3.1 The matrix \mathbb{V}_{bw}

The system of linear equations \mathbb{V}_{bw} is in the variables $(a_m)_{m \in N_{bw} \cap M}$ where $N_{bw} \subset M_{\mathbb{R}}$ is a convex polygon, introduced in Section 3.1.1.2, and called the *small Newton polygon*. There is one system for every pair $(b, w) \in B \times W$. The system \mathbb{V}_{bw} is of the form $(matrix)(a_m) = 0$; we also denote this matrix by \mathbb{V}_{bw} . Therefore, the columns of the matrix \mathbb{V}_{bw} are indexed by the lattice points $N_{bw} \cap M$. By Corollary 5.3, the polygon N_{bw} is the Newton polygon of the Laurent polynomial Q_{bw} .

The equations in \mathbb{V}_{bw} , i.e., the rows of the matrix \mathbb{V}_{bw} are defined in Section 3.1.2. There are two types:

- 1. There is a row for each of the points $(p_1, q_1), \ldots, (p_g, q_g)$ of the divisor S on the spectral curve. The entry of the row in column $m \in N_{\text{bw}} \cap M$ is $\chi^m(p_i, q_i)$.
- 2. The remaining rows correspond to certain zig-zag paths α . The entries in the row corresponding to α are certain monomials in C_{α} .

Let us proceed to the precise definition of the matrix \mathbb{V}_{bw} .

3.1.1 Columns of the matrix \mathbb{V}_{bw}

We now describe the small Newton polygons, whose lattice points correspond to columns of \mathbb{V}_{bw} .

3.1.1.1 Rational Abel map D. Recall the set $\{D_{\rho}\}$ of lines at infinity of the toric surface X_N . Consider the Q-vector space $\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}(X_N)$ of Q-divisors at infinity, defined as the Q-vector space with a basis given by the divisors D_{ρ} :

$$\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}(X_N) := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Q}D_{\rho}.$$

We define a rational Abel map

$$\mathbf{D}: V \to \operatorname{Div}^{\mathbb{Q}}_{\mathrm{T}}(X_N)$$

which assigns to each vertex v of the graph Γ a Q-divisor at infinity $\mathbf{D}(v)$ as follows:

- 1. Normalize $\mathbf{D}(\mathbf{w}) = 0$. As in the case of \mathbf{d} , the choice of normalization plays no role, and we can replace 0 with any \mathbb{Q} -divisor.
- 2. For any path γ contained in R from v_1 to v_2 ,

$$\mathbf{D}(\mathbf{v}_2) - \mathbf{D}(\mathbf{v}_1) = \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}} \frac{(\alpha, \gamma)_R}{|E_{\rho}|} D_{\rho},$$

where $(\cdot, \cdot)_R$ is the intersection index in R, i.e., the signed number of intersections of α with γ .

The following lemma follows from definitions.

Lemma 3.2. Let α, β be the zig-zag paths through e = bw, with $\alpha \in Z_{\sigma}, \beta \in Z_{\rho}$. Then, we have

$$\mathbf{D}(\mathbf{w}) - \mathbf{D}(\mathbf{b}) = -\frac{1}{|E_{\sigma}|} D_{\sigma} - \frac{1}{|E_{\rho}|} D_{\rho} - \operatorname{div} \phi(e)$$
(18)

where $\phi(e)$ is the character defined in (1) and div $\phi(e)$ denotes its (Weil) divisor as in (48).

3.1.1.2 Small Newton polygons. Recall the divisor D_N at infinity of X_N , see (6). Given an edge e = bw, we define a Q-divisor at infinity

$$Y_{\rm bw} := D_N - \mathbf{D}(w) + \mathbf{D}(b) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: b \in \alpha} \frac{1}{|E_\rho|} D_\rho.$$
(19)

Here the double sum is over all zig-zag paths α passing through b and $|E_{\rho}|$ denotes the number of integral primitive vectors in E_{ρ} as in Section 2.6. We define $b_{\rho} \in \mathbb{Q}$ as the multiplicities of the projective lines at infinity D_{ρ} in the divisor Y_{bw} :

$$Y_{\rm bw} = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}.$$
 (20)



Figure 6: The two small polygons in Example 3.4. The big black dot denotes the origin, while the other black dots are integral points.

Definition 3.3. The small Newton polygon $N_{\rm bw}$ is the polygon defined by the formula

$$N_{\rm bw} = \bigcap_{\rho \in \Sigma(1)} \{ m \in \mathcal{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle \ge -b_{\rho} \}.$$
⁽²¹⁾

There is a canonical bijection between divisors D in $\operatorname{Div}_{\mathrm{T}}^{\mathbb{Q}}(X_N)$ and convex polygons P with rational intercepts (see Proposition A.3 for its importance in toric geometry):

$$D = \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho} \quad \leftrightarrow \quad P = \bigcap_{\rho \in \Sigma(1)} \{ m \in \mathcal{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle \ge -a_{\rho} \}, \qquad a_{\rho} \in \mathbb{Q}.$$
(22)

Therefore, $N_{\rm bw}$ is the polygon associated to the divisor $Y_{\rm bw}$ in (22).

The polygon $N_{\rm bw}$ may not be integral. We will consider only integral points in it. The convex hull of the integral points in $N_{\rm bw}$ contains the Newton polygon of $Q_{\rm bw}$ (Corollary 5.3).

Example 3.4. We compute the small polygons for the square lattice in Figure 5. Recall that we chose $\mathbf{w} = w_1$. Since there is only one zig-zag path in each homology direction, the rational Abel map **D** is obtained from **d** by replacing the point at infinity with the corresponding line at infinity, so from Example 2.6, we have

$$\mathbf{D}(\mathbf{b}_1) = D_{\rho(\gamma)}, \quad \mathbf{D}(\mathbf{b}_2) = D_{\rho(\alpha)}, \quad \mathbf{D}(\mathbf{w}_1) = -D_{\rho(\beta)}, \quad \mathbf{D}(\mathbf{w}_2) = -D_{\rho(\delta)}.$$

We have $D_N = D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}$, using which we compute

$$\begin{split} Y_{b_{1}w_{1}} &= (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) - (-D_{\rho(\beta)}) + D_{\rho(\gamma)} - (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) \\ &= D_{\rho(\beta)} + D_{\rho(\gamma)}, \\ Y_{b_{2}w_{1}} &= (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) - (-D_{\rho(\beta)}) + D_{\rho(\alpha)} - (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) \\ &= D_{\rho(\alpha)} + D_{\rho(\beta)}. \end{split}$$

Therefore,

$$\begin{split} N_{\mathbf{b}_1\mathbf{w}_1} &= \{-i-j \ge 0\} \cap \{i-j \ge -1\} \cap \{i+j \ge -1\} \cap \{-i+j \ge 0\},\\ N_{\mathbf{b}_2\mathbf{w}_1} &= \{-i-j \ge -1\} \cap \{i-j \ge -1\} \cap \{i+j \ge 0\} \cap \{-i+j \ge 0\}, \end{split}$$

see Figure 6. Note that the convex hulls of the lattice points are the Newton polygons of $Q_{b_1w_1}$ and $Q_{b_2w_1}$ in (15).

3.1.2 Rows of the matrix \mathbb{V}_{bw}

Recall that the variables in \mathbb{V}_{bw} are $(a_m)_{m \in N_{bw} \cap M}$. We identify a Laurent polynomial $F = \sum_{m \in M} b_m \chi^m$ with its vector of coefficients $(b_m)_{m \in M}$. The equations in \mathbb{V}_{bw} are of two types:

1. For each $1 \leq i \leq g$, we have the linear equations

$$\sum_{m \in N_{\rm bw} \cap \mathcal{M}} a_m \chi^m(p_i, q_i) = 0, \qquad (23)$$

so the entry of the corresponding row of \mathbb{V}_{bw} in column *m* is $\chi^m(p_i, q_i)$.

2. Recall the notation $\lfloor x \rfloor$ for the largest integer n such that $n \leq x$.

Given a Q-divisor $D = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$, we define a divisor with integral coefficients

$$\lfloor D \rfloor := \sum_{\rho \in \Sigma(1)} \lfloor b_\rho \rfloor D_\rho$$

Recall the divisor Y_{bw} in (19). For a divisor D at infinity, let $D|_{\mathcal{C}}$ denote the divisor corresponding to the intersection of D with \mathcal{C} . Precisely, if $D = \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}$, then $D|_{\mathcal{C}} := \sum_{\rho \in \Sigma(1)} a_{\rho} \sum_{\alpha \in Z_{\rho}} \nu(\alpha)$. We have a linear equation for every zig-zag path α such that $\nu(\alpha)$ appears in

$$-D_N \big|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} \nu(\alpha) + \lfloor Y_{\mathbf{bw}} \rfloor \big|_{\mathcal{C}}.$$
 (24)

Suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes an equation. We extend $[\alpha]$ to a basis (x_1, x_2) of M, where $x_1 := [\alpha]$ and $\langle x_2, u_{\rho} \rangle = 1$, so that for any $m \in M$, we can write

$$\chi^m = x_1^{b_m} x_2^{c_m}, \quad b_m, c_m \in \mathbb{Z}$$

Let $N_{\rm bw}^{\rho}$ be the set of lattice points in $N_{\rm bw}$ closest to the edge E_{ρ} of N i.e., the set of points in $N_{\rm bw}$ that minimize the functional $\langle *, u_{\rho} \rangle$. Then the equation associated with α is

$$\sum_{a \in N_{\rm bw}^{\rho} \cap \mathcal{M}} a_m C_{\alpha}^{-b_m} = 0.$$
⁽²⁵⁾

So the entry in column $m \in N_{\text{bw}}^{\rho} \cap M$ is the monomial $C_{\alpha}^{-b_m}$, and the entries in the other columns are 0. Choosing a different basis vector x_2 leads to the same equation multiplied by a monomial in C_{α} .

Remark 3.5. When all the sides of the Newton polygon are primitive, we call the Newton polygon simple. In this case, we have $[Y_{bw}] = Y_{bw}$ and $\mathbf{d}(\mathbf{w}) - \mathbf{d}(b) = (\mathbf{D}(\mathbf{w}) - \mathbf{D}(b))|_{\mathcal{C}}$. Then Formula (24) simplifies considerably to

m

$$\sum_{\substack{\in Z: b \notin \alpha}} \nu(\alpha). \tag{26}$$

So for a simple Newton polygon the Casimir rows of the matrix \mathbb{V}_{bw} , i.e., the rows providing equations (25), are parameterized by the zig-zag paths α which do not contain the vertex b.

3.1.3 The functions V_{bw}

The number of rows of \mathbb{V}_{bw} is at least as large as the number of columns minus one, but not necessarily equal. However, Proposition 7.3 shows that there is a unique solution to \mathbb{V}_{bw} up to a multiplicative constant. Therefore,

$$V_{\rm bw} := \sum_{m \in N_{\rm bw} \cap \mathcal{M}} a_m \chi^m, \tag{27}$$

is uniquely defined up to a multiplicative constant (where $(a_m)_{m \in N_{bw} \cap M}$ is a solution to \mathbb{V}_{bw}). Only ratios of the values of these functions that are independent of the multiplicative constant appear in the inverse map, see Section 3.2.

Remark 3.6. When the equations in \mathbb{V}_{bw} are linearly independent (so there is exactly one less equation than the number of variables), we can prepend to \mathbb{V}_{bw} the equation $\sum_{m \in N_{bw} \cap M} a_m \chi^m$ to get a square matrix, which we denote by \mathbb{V}_{bw}^{χ} . Then the function V_{bw} is the determinant:

$$V_{bw} = \det \mathbb{V}_{bw}^{\chi}$$

Indeed, given an $(n-1) \times n$ matrix (a_{ij}) , the system of linear equations $\sum_{j=1}^{n} a_{ij} x_j = 0$ has a solution given by the signed maximal minors A_j of the matrix A:

$$x_j = (-1)^j A_j.$$

Here A_j is the determinant of the matrix obtained by deleting the *j*-th column of A. Therefore, the determinant of the augmented matrix $\mathbb{V}_{\text{bw}}^{\chi}$ recovers the expression V_{bw} in (27).

Example 3.7. We compute the linear system of equations \mathbb{V}_{bw} for the square lattice in Figure 5 with $\mathbf{w} = \mathbf{w}_1$. Since both black vertices are contained in every zig-zag path, the formula (26) is 0, so there are no equations of type 2 in \mathbb{V}_{bw} for $\mathbf{b} \in B$. Therefore,

$$\mathbb{V}_{\mathbf{b}_1\mathbf{w}} = \begin{pmatrix} 1 & p^{-1} \end{pmatrix}, \quad \mathbb{V}_{\mathbf{b}_2\mathbf{w}} = \begin{pmatrix} 1 & q \end{pmatrix}.$$

By Remark 3.6, we get

$$\mathbf{V}_{\mathbf{b}_1 \mathbf{w}} = \begin{vmatrix} 1 & z^{-1} \\ 1 & p^{-1} \end{vmatrix}, \quad \mathbf{V}_{\mathbf{b}_2 \mathbf{w}} = \begin{vmatrix} 1 & w \\ 1 & q \end{vmatrix}.$$
(28)

Using (16), we have

$$\kappa_{\Gamma,\mathbf{w}}^*(\mathcal{V}_{\mathbf{b}_1\mathbf{w}}) = AX_1X_2 - \frac{1}{z} = AX_2Q_{\mathbf{b}_1\mathbf{w}}$$
$$\kappa_{\Gamma,\mathbf{w}}^*(\mathcal{V}_{\mathbf{b}_2\mathbf{w}}) = \frac{1}{B} - w = \frac{1}{B}Q_{\mathbf{b}_2\mathbf{w}},$$

verifying the conclusion of Theorem 3.1.

3.2 Reconstructing weights via functions V_{bw} .

Take a white vertex w and a zig-zag path α containing w. The pair (w, α) determines a wedge $W := b \xrightarrow{e} w \xrightarrow{e'} b'$, where w is a white vertex incident to the vertices b, b' such that bwb' is a part of α . Recall $\phi(e)$ from (1), and the Kasteleyn sign $\epsilon(e)$. We assign to this wedge the ratio

$$r_W := -\frac{\epsilon(e')\phi(e')V_{b'\mathbf{w}}}{\epsilon(e)\phi(e)V_{b\mathbf{w}}}(\nu(\alpha)).$$
(29)

Note that we use the distinguished white vertex \mathbf{w} in the expression rather than w. The expression is in fact independent of w, as we will see in the proof of Theorem 3.10 below.

Remark 3.8. The ratio on the right is a rational function on the curve. We evaluate the ratio at the point at infinity of the spectral curve $\nu(\alpha)$ corresponding to the zig-zag path α , see (11). To do this, we first extend $[\alpha]$ to a basis (x_1, x_2) of M with $[\alpha] = x_1$ and $\langle x_2, u_{\rho} \rangle = 1$, as explained in Section 2.7. Then $\nu(\alpha)$ is given by $\frac{1}{x_1} = C_{\alpha}, x_2 = 0$. The numerator and denominator in (29) vanish to the same order in x_2 by Corollary 6.2 below, so after factoring out and canceling the highest power of x_2 in the numerator and denominator, we can evaluate at $x_1 = \frac{1}{C_{\alpha}}, x_2 = 0$ to get a well-defined number.

Let $L = b_1 \to w_1 \to b_2 \to \cdots \to b_\ell = b_1$ be an oriented loop on Γ . It is a concatenation of wedges $W_i := b_{i-1} w_i b_i, i = 1, \dots, \ell$ (with *i* taking values cyclic modulo ℓ) provided by the white vertices. Denote by α_i the zig-zag path assigned to the wedge W_i . We define a cohomology class $[\omega]$ by

$$[\omega]([L]) := \prod_{i=1}^{\ell} r_{W_i}.$$
(30)

Lemma 3.9. The product (30) does not depend on the ambiguities of the multiplicative constants in the involved functions V_{bw} .

Proof. For each black vertex b_i in L, $V_{b_i w}$ appears twice in (30), once each in the numerator and denominator, and so the multiplicative constants cancel out.

Theorem 3.10. The cohomology classes [wt] and $\kappa^*_{\Gamma,\mathbf{w}}[\omega]$ are equal.

Proof. Let $b \xrightarrow{e} w \xrightarrow{e'} b'$ be a wedge with zig-zag path $\alpha \in Z_{\rho}$. The restriction of the characteristic polynomial $P(z, w)|_{D_{\rho}}$ is the partition function of those dimers whose homology class in N lies on E_{ρ} . From the explicit construction of external dimers in [GK13] (that is, dimers whose homology classes are in ∂N), we have that each dimer with homology class in E_{ρ} uses exactly one of the edges e or e'. Since $Q_{bw}(z, w)$ is the partition function of dimers with the vertices b, w removed, we have

$$P\big|_{D_{\rho}} = wt(e)\epsilon(e)\phi(e)Q_{\mathrm{bw}}\big|_{D_{\rho}} + wt(e')\epsilon(e')\phi(e')Q_{\mathrm{b'w}}\big|_{D_{\rho}}.$$

Since $\nu(\alpha)$ is on the spectral curve, $P(\nu(\alpha)) = 0$, from which we get

$$\frac{wt(e)}{wt(e')} = -\frac{\epsilon(e')\phi(e')Q_{\mathbf{b'w}}}{\epsilon(e)\phi(e)Q_{\mathbf{bw}}}(\nu(\alpha)).$$
(31)

We have $\operatorname{corank}(K) = 1$ at smooth points of \mathcal{C} . Note that $KQ|_{\mathcal{C}} = 0$. Therefore, for generic wt, since \mathcal{C} is smooth, Q is a rank 1 matrix given by

$$Q = \ker K^* \otimes \operatorname{coker} K.$$

This implies that

$$\frac{Q_{\rm bw}}{Q_{\rm b'w}}(\nu(\alpha)) = \frac{Q_{\rm bw}}{Q_{\rm b'w}}(\nu(\alpha)).$$

Example 3.11. Consider the cycle *a* in Figure 5 given by the red horizontal path. We write it as the concatenation of the two wedges W_1 and W_2 represented by (w_1, δ) and (w_1, γ) respectively. From

Table (10), we know that in the basis $x_1 = zw, x_2 = w$, the point $\nu(\delta)$ is given by $x_1 = \frac{1}{C_{\delta}}, x_2 = 0$. Using (28), and making the substitution $z = \frac{x_1}{x_2}, w = x_2$, we get

$$r_{W_1} = -\frac{-1 \cdot w^{-1} \cdot V_{b_2 \mathbf{w}}}{-1 \cdot z \cdot V_{b_1 \mathbf{w}}} (\nu(\delta))$$
$$= -\frac{1}{zw} \frac{q - w}{p^{-1} - z^{-1}} (\nu(\delta))$$
$$= \frac{-(q - x_2)}{x_1 p^{-1} - x_2} \left(\frac{1}{C_{\delta}}, 0\right)$$
$$= -pqC_{\delta}.$$

Similarly, from table (10) we know that in the basis $x_1 = \frac{z}{w}, x_2 = w$, the point $\nu(\gamma)$ is given by $x_1 = \frac{1}{C_{\gamma}}, x_2 = 0$. Using (28), and making the substitution $z = x_1 x_2, w = x_2$, we get

$$r_{W_2} = -\frac{1 \cdot 1 \cdot V_{b_1 \mathbf{w}}}{-1 \cdot w^{-1} \cdot V_{b_2 \mathbf{w}}} (\nu(\gamma))$$
$$= w \frac{p^{-1} - z^{-1}}{q - w} (\nu(\gamma))$$
$$= x_2 \frac{p^{-1} - \frac{1}{x_1 x_2}}{q - x_2} \left(\frac{1}{C_{\gamma}}, 0\right)$$
$$= -\frac{C_{\gamma}}{q}.$$

Therefore, $[\omega]([a]) = pC_{\gamma}C_{\delta}$, and using (9) and (16), we have

$$\kappa_{\Gamma,\mathbf{w}}^*[\omega]([a]) = \left(\frac{1}{AX_1X_2}\right) \cdot \left(-\frac{AX_1X_2X_3}{B}\right) \cdot \left(-\frac{AB}{X_3}\right)$$
$$= A.$$

4 Examples

In this section, we work out two detailed examples.

4.1 Primitive genus 2 example

Consider the hexagonal graph Γ with Newton polygon N and normal fan Σ as shown in Figure 7. We label the vertices of Γ as in Figure 8. We label the zig-zag paths by α, β, γ , and denote the ray of Σ dual to $\tau \in \{\alpha, \beta, \gamma\}$ by σ_{τ} .

We can take $X_i = [wt]([\partial f_i]), i = 1, ..., 4$, and A = [wt]([a]), B = [wt]([b]) as coordinates on $H^1(\Gamma, \mathbb{C})$ (see Figure 8).

The Casimirs are

$$C_{\alpha} = -\frac{B^2 X_1 X_2 X_4}{A}, \quad C_{\beta} = -\frac{X_3}{A B^3 X_1^2 X_4}, \quad C_{\gamma} = \frac{A^2 B X_1}{X_2 X_3}.$$
 (32)



Figure 7: A hexagonal graph, its Newton polygon N and normal fan Σ , with zig-zag paths and rays labeled.



Figure 8: Labeling of the vertices and faces of Γ , a cocycle representing [wt] and ϕ , where $X_i = [wt]([\partial f_i]), A = [wt]([a]), B = [wt](b)$, and a and b are the red and green cycles respectively. The Kasteleyn sign ϵ is 1 for all edges. If no weight or ϕ is indicated for an edge, it means that it is 1.



Figure 9: The small polygons for the hexagonal graph.

The Kasteleyn matrix is

$$K = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ 1 & 0 & 1 & 0 & Az \\ \frac{1}{X_2} & X_3 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{ABX_1zw} & 0 & 1 \\ X_1X_4Bw & 0 & 1 & 1 & 0 \\ 0 & Bw & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}$$

Let $P(z, w) = \det K$ and $C = \overline{\{P(z, w) = 0\}}$. The spectral transform is $\kappa_{\Gamma, \mathbf{w}} = (C, S, \nu) \in S_N$, where since the interior of N contains two lattice points, the divisor $S = (p_1, q_1) + (p_2, q_2)$ is a sum

of two points, where

$$p_{1} = -\frac{\sqrt{(-BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} - B)^{2} - 4B^{2}X_{1}X_{2}X_{4}} + BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} + B}{2ABX_{1}},$$

$$q_{1} = \frac{-\sqrt{(-BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} - B)^{2} - 4B^{2}X_{1}X_{2}X_{4}} + BX_{1}X_{2}X_{3}X_{4} + BX_{1}X_{2}X_{4} + B}{2B^{2}X_{1}X_{2}X_{4}}},$$

$$p_{2} = -\frac{-\sqrt{(-BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} - B)^{2} - 4B^{2}X_{1}X_{2}X_{4}} + BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} + B}{2ABX_{1}}}{2ABX_{1}},$$

$$q_{2} = \frac{\sqrt{(-BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} - B)^{2} - 4B^{2}X_{1}X_{2}X_{4}} + BX_{1}X_{2}X_{3}X_{4} - BX_{1}X_{2}X_{4} + B}{2B^{2}X_{1}X_{2}X_{4}} + BX_{1}X_{2}X_{3}X_{4} + BX_{1}X_{2}X_{4} + B}}{2B^{2}X_{1}X_{2}X_{4}},$$

$$(33)$$

The points at infinity are given by the following table:

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity	
α	(-1, 2)	(-1, 2), (0, -1)	$x_1 = \frac{1}{C_\alpha}, x_2 = 0$	(24)
β	(-1, -3)	(-1, -3), (0, -1)	$x_1 = \frac{1}{C_\beta}, x_2 = 0$	(34)
γ	(2, 1)	(2,1), (-1,0)	$x_1 = \frac{1}{C_{\gamma}}, x_2 = 0$	

The discrete Abel map \mathbf{D} is given by

$$\begin{aligned} \mathbf{D}(\mathbf{w}) &= 0, & \mathbf{D}(\mathbf{b}_1) = D_\beta + D_\gamma, & \mathbf{D}(\mathbf{b}_2) = -D_\alpha + 2D_\beta + D_\gamma, \\ \mathbf{D}(\mathbf{b}_3) &= D_\alpha + D_\beta, & \mathbf{D}(\mathbf{b}_4) = 2D_\beta, & \mathbf{D}(\mathbf{b}_5) = -D_\alpha + 3D_\beta, \end{aligned}$$

and $D_N = 2D_{\alpha} + 2D_{\beta} + D_{\gamma}$. Since $\mathbf{D}(\mathbf{w}) = 0$ and every black vertex b is contained in every zig-zag path, we have

$$Y_{\mathbf{b}\mathbf{w}} = 2D_{\alpha} + 2D_{\beta} + D_{\gamma} + \mathbf{D}(\mathbf{b}) - D_{\alpha} - D_{\beta} - D_{\gamma}$$
$$= \mathbf{D}(\mathbf{b}) + D_{\alpha} + D_{\beta}.$$

Using this, we compute

$$\begin{split} Y_{\mathbf{b}_1\mathbf{w}} &= D_{\alpha} + 2D_{\beta} + D_{\gamma}, \qquad Y_{\mathbf{b}_2\mathbf{w}} = 3D_{\beta} + D_{\gamma}, \qquad Y_{\mathbf{b}_3\mathbf{w}} = 2D_{\alpha} + 2D_{\beta}, \\ Y_{\mathbf{b}_4\mathbf{w}} &= D_{\alpha} + 3D_{\beta}, \qquad Y_{\mathbf{b}_5\mathbf{w}} = 4D_{\beta}. \end{split}$$

The small polygons are shown in Figure 9. Since the Newton polygon N is primitive, we are in the setting of Remark 3.5. Every zig-zag path contains every black vertex, so the expression (26) is 0. Therefore, there are no equations of type 2 in the linear system \mathbb{V}_{bw} for any black vertex b. Since g = 2, we have two equations of type 1 for every black vertex b. Moreover, we note that each of the small polygons in Figure 9 contains exactly three lattice points, so by Remark 3.6, we get

$$\mathbf{V}_{\mathbf{b}_{1}\mathbf{w}} = \begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}, \qquad \mathbf{V}_{\mathbf{b}_{2}\mathbf{w}} = \begin{vmatrix} 1 & z^{-1} & z^{-1}w^{-1} \\ 1 & p_{1}^{-1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & p_{2}^{-1} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}, \qquad \mathbf{V}_{\mathbf{b}_{3}\mathbf{w}} = \begin{vmatrix} 1 & w & w^{2} \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix},$$

$$\mathbf{V}_{\mathbf{b}_{4}\mathbf{w}} = \begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}, \qquad \mathbf{V}_{\mathbf{b}_{5}\mathbf{w}} = \begin{vmatrix} 1 & z^{-1}w & z^{-1} \\ 1 & p_{1}^{-1}q_{1} & p_{1}^{-1} \\ 1 & p_{2}^{-1}q_{2} & p_{2}^{-1} \end{vmatrix}.$$

The boundary of the face f_2 is the concatenation of the three wedges W_1, W_2 and W_3 represented by $(w_2, \alpha), (\mathbf{w}, \beta)$ and (w_4, γ) respectively. We compute

$$r_{W_{1}} = -\frac{V_{b_{1}w_{2}}}{V_{b_{4}w_{2}}}(\nu(\alpha)) = -\frac{\begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}}(\nu(\alpha)).$$

To evaluate at $\nu(\alpha)$, as explained in Remark 3.8, we extend $[\alpha] = (-1, 2)$ to the basis (x_1, x_2) of M, where $x_1 = [\alpha] = (-1, 2)$ and $x_2 = (0, -1)$. Then $\nu(\alpha)$ is given by $x_1 = \frac{1}{C_{\alpha}}, x_2 = 0$. Expressing z, w in the basis (x_1, x_2) as $z = \frac{1}{x_1 x_2^2}, w = \frac{1}{x_2}$, we get

$$r_{W_{1}} = -\frac{\begin{vmatrix} 1 & \frac{1}{x_{2}} & x_{1}x_{2}^{2} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{x_{2}} & x_{1}x_{2}^{2} \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}} \left(\frac{1}{C_{\alpha}}, 0\right) = -\frac{\begin{vmatrix} x_{2} & 1 & x_{1}x_{2}^{4} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}{\begin{vmatrix} x_{2} & 1 & x_{1}x_{2}^{3} \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}} \left(\frac{1}{C_{\alpha}}, 0\right) = -\frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}}$$
$$= -\frac{p_{1}q_{1} - p_{2}q_{2}}{q_{1}q_{2}(p_{1} - p_{2})},$$

where we factored out x_2 from the numerator and denominator and then evaluated at $(x_1, x_2) = (\frac{1}{C_{\alpha}}, 0)$.

For W_2 , letting $(x_1, x_2) = ((-1, -3), (0, -1))$ we have $z = \frac{x_2^3}{x_1}, w = \frac{1}{x_2}$, and $\nu(\beta)$ is given by $x_1 = \frac{1}{C_{\beta}}, x_2 = 0$. Therefore, we get

$$r_{W_{2}} = -\frac{V_{b_{3}\mathbf{w}}}{V_{b_{1}\mathbf{w}}}(\nu(\beta)) = -\frac{\begin{vmatrix} 1 & w & w^{2} \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix}}{\begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}(\nu(\beta)) = -\frac{\begin{vmatrix} 1 & \frac{1}{x_{2}} & \frac{1}{x_{2}^{2}} \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{x_{2}} & \frac{x_{1}}{x_{2}^{2}} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}\left(\frac{1}{C_{\beta}}, 0\right) = -\frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & \frac{1}{C_{\beta}} \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}} = -C_{\beta}.$$

Finally, for W_3 , letting $(x_1, x_2) = ((2, 1), (-1, 0))$ we have $z = \frac{1}{x_2}, w = x_1 x_2^2$, and $\nu(\gamma)$ is given by

 $x_1 = \frac{1}{C_{\gamma}}, x_2 = 0.$ Therefore, we get

$$r_{W_3} = -\frac{V_{b_4 \mathbf{w}}}{V_{b_3 \mathbf{w}}}(\nu(\gamma)) = -\frac{\begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & w & w^2 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}(\nu(\gamma)) = -\frac{\begin{vmatrix} 1 & x_1 x_2^2 & x_2 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 x_2^2 & x_1^2 x_2^4 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}\left(\frac{1}{C_{\gamma}}, 0\right) = -\frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}$$
$$= \frac{p_1 q_1 - p_2 q_2}{p_1 p_2 q_1 q_2 (q_1 - q_2)}.$$

Putting everything together, we get

$$X_{2} = -\frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_{1} & p_{1}^{-1}q_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1}q_{2}^{-1} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}} \begin{vmatrix} 0 & 0 & 1 \\ 1 & q_{1} & q_{1}^{2} \\ 1 & q_{2} & q_{2}^{2} \end{vmatrix}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & q_{1} & p_{1}^{-1} \\ 1 & q_{2} & p_{2}^{-1} \end{vmatrix}} \\= \frac{C_{\beta}(p_{1}q_{1} - p_{2}q_{2})^{2}}{p_{1}p_{2}q_{1}^{2}q_{2}^{2}(p_{1} - p_{2})(q_{1} - q_{2})},$$

with similar formulas for X_1, X_3, X_4, A, B . It may be easily verified that these invert the spectral transform by plugging in the formulas (32) and (33) into the right-hand side and simplifying using computer algebra.

4.2 Non-primitive example

Consider the square-octagon graph Γ with Newton polygon N and normal fan Σ as shown in Figure 10. We label the vertices of Γ as in Figure 11. We label the rays of Σ by $\sigma_{\alpha}, \sigma_{\beta}, \sigma_{\gamma}, \sigma_{\delta}$ and the two zig-zag paths dual to ray σ_{τ} by $\{\tau_1, \tau_2\}$, for $\tau \in \{\alpha, \beta, \gamma, \delta\}$.

We can take $X_i := [wt]([\partial f_i]), i = 1, ..., 7$, and A := [wt]([a]), B := [wt]([b]) as coordinates on $H^1(\Gamma, \mathbb{C})$ (see Figure 11). The Casimirs are

$$\begin{aligned} C_{\alpha_1} &= X_1 X_3 X_7 B, \quad C_{\alpha_2} = \frac{B X_2 X_3 X_4 X_6 X_7}{X_1 X_5}, \quad C_{\beta_1} = \frac{X_2}{A X_1 X_5}, \quad C_{\beta_2} = \frac{1}{A X_7}, \\ C_{\gamma_1} &= \frac{X_5}{B X_1 X_3}, \quad C_{\gamma_2} = \frac{X_6}{B}, \quad C_{\delta_1} = \frac{A X_1}{X_2 X_6}, \quad C_{\delta_2} = \frac{A X_1 X_5}{X_2 X_3 X_4 X_6 X_7}. \end{aligned}$$

Since the Newton polygon N has only one interior lattice point, the divisor S = (p, q) consists of a



Figure 10: A square-octagon graph, its Newton polygon N and normal fan Σ , with zig-zag paths and rays labeled.

single point. The Kasteleyn matrix is

$$K = \begin{pmatrix} 1 & 1 & 0 & Az & 0 & 0 & 0 & 0 \\ 1 & -X_7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{Bw} \\ 0 & 0 & \frac{X_1 X_5}{X_2 X_3 X_4 X_6 X_7} & -1 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{X_3} & 0 & 0 & 1 & \frac{1}{X_5} & 0 & 0 \\ Bw X_1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{X_1}{X_2} & X_6 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{Az} & 0 & 1 & -1 \end{pmatrix}$$

Let $P(z, w) = \det K$ and $C = \overline{\{P(z, w) = 0\}}$. The spectral transform is $\kappa_{\Gamma, \mathbf{w}} = (C, S, \nu) \in S_N$, where

$$p = -\frac{X_2 X_4 X_6 \left(X_3 X_5 X_6 X_7 \left(X_1^2 (X_4+1)+X_2 X_4\right)+X_1 X_2 X_3^2 X_4 X_6^2 X_7^2+X_1 X_5^2\right)}{A (X_1 X_5+X_2 X_3 X_4 X_6 X_7)} \\ \times \frac{1}{\left(X_3 X_4 X_6 X_7 \left(X_1^2 X_5+X_2 (X_5+1) (X_6+1)\right)+X_1 X_5 (X_6 (X_4+X_5+1)+X_5+1)\right)},$$

$$q = \frac{X_5 \left(-X_3 X_4 X_6 X_7 \left(X_1^2+X_2 X_6+X_2\right)-X_1 X_5 (X_6+1)\right)}{B X_1 X_3 X_7 (X_1 X_5 (X_4 X_6+X_6+1)+X_2 X_3 X_4 X_6 (X_6+1) X_7)}.$$

The table below lists the points at infinity for each of the zig-zag paths:



Figure 11: Labeling of the vertices and faces of Γ , and a cocycle and Kasteleyn sign, where $X_i = [wt]([\partial f_i]), A = [wt]([a]), B = [wt](b)$ and $U = \frac{X_1X_5}{X_2X_3X_4X_6X_7}$. The edges with no weight indicated have weight 1.

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity
α_1	(0.1)	(0,1), $(-1,0)$	$x_1 = \frac{1}{C_{\alpha_1}}, x_2 = 0$
α_2	(0,2)	(0,1),(1,0)	$x_1 = \frac{1}{C_{\alpha_2}}, x_2 = 0$
β_1	(-1.0)	(-10)(0-1)	$x_1 = \frac{1}{C_{\beta_1}}, x_2 = 0$
β_2	(-1,0)	(-1,0),(0,-1)	$x_1 = \frac{1}{C_{\beta_2}}, x_2 = 0$
γ_1	(0 -1)	(0 - 1) (1 0)	$x_1 = \frac{1}{C_{\gamma_1}}, x_2 = 0$
γ_2	(0,-1)	(0, 1),(1,0)	$x_1 = \frac{1}{C_{\gamma_2}}, x_2 = 0$
δ_1	(1.0)	(1 0) (0 1)	$x_1 = \frac{1}{C_{\delta_1}}, x_2 = 0$
δ_2	(1,0)	(1,0),(0,1)	$x_1 = \frac{1}{C_{\delta_2}}, x_2 = 0$



Figure 12: The small polygons for the square-octagon graph.

The discrete Abel map **D** is given by $\mathbf{D}(\mathbf{w}) = 0$ and

$$\begin{aligned} \mathbf{D}(\mathbf{b}_{1}) &= \frac{1}{2}D_{\gamma} + \frac{1}{2}D_{\delta}, \\ \mathbf{D}(\mathbf{b}_{3}) &= \frac{1}{2}(-D_{\alpha} + D_{\beta} + D_{\gamma} + D_{\delta}), \\ \mathbf{D}(\mathbf{b}_{5}) &= D_{\beta}, \\ \mathbf{D}(\mathbf{b}_{5}) &= -\frac{1}{2}D_{\alpha} + \frac{1}{2}D_{\beta} + D_{\gamma}, \end{aligned} \qquad \begin{aligned} \mathbf{D}(\mathbf{b}_{6}) &= -\frac{1}{2}D_{\alpha} + D_{\beta} + \frac{1}{2}D_{\gamma} \\ \mathbf{D}(\mathbf{b}_{7}) &= -\frac{1}{2}D_{\alpha} + \frac{1}{2}D_{\beta} + D_{\gamma}, \end{aligned} \qquad \begin{aligned} \mathbf{D}(\mathbf{b}_{8}) &= -\frac{1}{2}D_{\alpha} + D_{\beta} + \frac{1}{2}D_{\gamma} \\ \mathbf{D}(\mathbf{b}_{7}) &= -\frac{1}{2}D_{\alpha} + \frac{1}{2}D_{\beta} + D_{\gamma}, \end{aligned}$$

We have $D_N = D_{\alpha} + D_{\beta} + D_{\gamma} + D_{\delta}$, using which we compute

$$\begin{split} Y_{b_{1}\mathbf{w}} &= \frac{1}{2}D_{\alpha} + D_{\beta} + D_{\gamma} + D_{\delta}, \\ Y_{b_{3}\mathbf{w}} &= D_{\beta} + \frac{3}{2}D_{\gamma} + D_{\delta}, \\ Y_{b_{5}\mathbf{w}} &= \frac{1}{2}D_{\alpha} + \frac{3}{2}D_{\beta} + D_{\gamma} + \frac{1}{2}D_{\delta}, \\ Y_{b_{7}\mathbf{w}} &= \frac{3}{2}D_{\beta} + \frac{3}{2}D_{\gamma} + \frac{1}{2}D_{\delta}, \end{split} \qquad \begin{aligned} Y_{b_{6}\mathbf{w}} &= \frac{1}{2}D_{\alpha} + \frac{3}{2}D_{\beta} + D_{\gamma} + \frac{1}{2}D_{\delta}, \\ Y_{b_{7}\mathbf{w}} &= \frac{3}{2}D_{\beta} + \frac{3}{2}D_{\gamma} + \frac{1}{2}D_{\delta}, \end{aligned} \qquad \begin{aligned} Y_{b_{8}\mathbf{w}} &= \frac{3}{2}D_{\beta} + \frac{3}{2}D_{\gamma} + \frac{1}{2}D_{\delta}. \end{aligned}$$

The corresponding small polygons are shown in Figure 12. Therefore, we have

$$V_{b_1 \mathbf{w}} = a_{(-1,-1)} z^{-1} w^{-1} + a_{(0,-1)} w^{-1} + a_{(1,-1)} z w^{-1} + a_{(-1,0)} z^{-1} + a_{(0,0)} + a_{(1,0)} z,$$

where the a_m satisfy the system of equations $\mathbb{V}_{b_1 \mathbf{w}}$ that we now determine. We have the equation of type 1:

$$a_{(-1,-1)}p^{-1}q^{-1} + a_{(0,-1)}q^{-1} + a_{(1,-1)}pq^{-1} + a_{(-1,0)}p^{-1} + a_{(0,0)} + a_{(1,0)}p = 0.$$

To find the zig-zag paths that contribute equations of type 2, we compute (24). We have

$$-D_{N}|_{\mathcal{C}} = \nu(\alpha_{1}) + \nu(\alpha_{2}) + \nu(\beta_{1}) + \nu(\beta_{2}) + \nu(\gamma_{1}) + \nu(\gamma_{2}) + \nu(\delta_{1}) + \nu(\delta_{2}),$$

$$\mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}_{1}) = -\nu(\gamma_{1}) - \nu(\delta_{2}),$$

$$\left[Y_{\mathbf{b}_{1}\mathbf{w}}\right]|_{\mathcal{C}} = \nu(\beta_{1}) + \nu(\beta_{2}) + \nu(\gamma_{1}) + \nu(\gamma_{2}) + \nu(\delta_{1}) + \nu(\delta_{2}),$$

using which we get that (24) is equal to $\nu(\beta_1) + \nu(\beta_2) + \nu(\gamma_2) + \nu(\delta_1)$, so we have four equations of type 2, one for each of the zig-zag paths $\beta_1, \beta_2, \gamma_2, \delta_1$.

Therefore, we have 5 equations and 6 variables, so we are in the setting of Remark 3.6 where $V_{b_1 \mathbf{w}} = \det \mathbb{V}_{b_1 \mathbf{w}}^{\chi}$. Computing the equations of type 2, we get

$$\mathbf{V}_{\mathbf{b}_{1}\mathbf{w}} = \begin{vmatrix} z^{-1}w^{-1} & w^{-1} & zw^{-1} & z^{-1} & 1 & z \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_{2}} & 0 & 0 & 0 & 0 \\ C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_{1}} & 1 \end{vmatrix}.$$

In like fashion, for V_{b_2w} , we have an equation of type 1 and four equations of type 2 for the zig-zag paths $\beta_1, \gamma_2, \delta_1, \delta_2$. We compute

$$\mathbf{V}_{\mathbf{b}_{2}\mathbf{w}} = \begin{vmatrix} z^{-1}w & w & z^{-1} & 1 & z^{-1}w^{-1} & w^{-1} \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_{1}} & 0 & 0 & 0 & 0 \\ C_{\gamma_{2}} & 0 & 1 & 0 & C_{\gamma_{2}}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_{1}} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_{2}} & 1 \end{vmatrix}.$$

We write the boundary of the face f_7 as the concatenation of the two wedges W_1 and W_2 represented by (\mathbf{w}, γ_1) and (\mathbf{b}_2, α_1) respectively. We have

$$r_{W_1} = \frac{V_{b_2 \mathbf{w}}}{V_{b_1 \mathbf{w}}}(\nu(\gamma_1)) = \frac{\begin{vmatrix} C_{\gamma_1} & 0 & 1 & 0 & C_{\gamma_1}^{-1} & 0 \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_2} & 1 \end{vmatrix}}{\begin{vmatrix} C_{\gamma_1} & 0 & 1 & 0 & C_{\gamma_1}^{-1} \\ 0 & 0 & 1 & 0 & C_{\gamma_1}^{-1} & 0 \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \end{vmatrix}},$$

where to evaluate at $\nu(\gamma_1)$, we use the basis x_1, x_2 from table (35). Similarly, we compute

$$r_{W_2} = -\frac{V_{b_1}\mathbf{w}}{V_{b_2}\mathbf{w}}(\nu(\alpha_1)) = -\frac{\begin{vmatrix} 0 & C_{\alpha_1}^{-1} & 0 & 1 & 0 & C_{\alpha_1} \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \end{vmatrix}}{\begin{vmatrix} 0 & C_{\alpha_1}^{-1} & 0 & 1 & 0 & C_{\alpha_1} \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_2} & 1 \end{vmatrix}}.$$

It can be verified using computer algebra that $X_7 = r_{W_1} r_{W_2}$.

5 The small polygons

In the remaining sections, we prove the results stated in Section 3. In order to invert the spectral transform, we want to first reconstruct the Q_{bw} , the entries of the **w**-column of the adjugate matrix, from the spectral data. To do this, we need to first find the Newton polygon of the Q_{bw} , which we call the small polygons and denote by N_{bw} . Explicitly, N_{bw} is the convex hull of homology classes of dimer covers of $\Gamma - \{b, w\}$. However, it appears difficult to describe N_{bw} in a direct combinatorial way. Instead, we will re-express the problem in terms of toric geometry. The key to doing this is an extension of the Kasteleyn matrix, which is a map of trivial sheaves on T, to a map of locally free sheaves on a compactification of T. We are led to consider a stacky toric surface \mathcal{X}_N instead of the toric surface X_N , because such an extension does not exist on X_N unless the polygon has only primitive sides.

The basics of stacky toric surfaces are recalled in detail in Appendix A.3. For the convenience of the reader we reproduce some notation.

Let Σ be the normal fan of N. There is a stacky fan $\Sigma = (\Sigma, \beta)$ where

$$\beta : \mathbb{Z}^{\Sigma(1)} \to \mathcal{M}^{\vee},$$
$$\delta_{\rho} \mapsto |E_{\rho}|u_{\rho},$$

where u_{ρ} is the primitive normal to E_{ρ} . We identify the set of rays $\Sigma(1)$ of the fan Σ with the components D_{ρ} of the divisor at infinity

$$\rho \leftrightarrow \tau_{\rho} = \mathbb{R}_{\geq 0} u_{\rho}.$$

We assign to Σ a smooth *toric Deligne-Mumford stack* \mathfrak{X}_N , which contains the torus T as a dense open subset.

We consider the stack rather than the toric surface since we construct an extension of the Kasteleyn operator to a compactification of the torus T in Lemma 5.1. There is no such extension on the toric surface when the Newton polygon is not simple, but there is one on the stack.

5.1Extension of the Kasteleyn operator

Define for each black vertex **b** the line bundle

$$\mathcal{E}_{\mathbf{b}} := \mathcal{O}_{\mathfrak{X}_N} \Big(\mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \Big),$$

and for each white vertex w, the line bundle

$$\mathcal{F}_{w} := \mathcal{O}_{\mathfrak{X}_{N}}(\mathbf{D}(w)).$$

Let

$$\mathcal{E} := \bigoplus_{\mathbf{b} \in B} \mathcal{E}_{\mathbf{b}}, \qquad \mathcal{F} := \bigoplus_{\mathbf{w} \in W} \mathcal{F}_{\mathbf{w}}.$$

They are locally free sheaves of the same rank #B = #W on \mathfrak{X}_N .

Proposition 5.1. The Kasteleyn operator K extends to a map of locally free sheaves on X_N :

$$K: \mathcal{E} \to \mathcal{F}. \tag{36}$$

Proof. By definition,

$$K_{\rm wb} = \sum_{e \in E(\Gamma) \text{ incident to b,w}} wt(e)\epsilon(e)\phi(e).$$

We need to show that for any edge e with vertices b, w, the character $\phi(e)$ is a global section of

$$\mathcal{H}om_{\mathfrak{X}_{N}}(\mathcal{E}_{\mathrm{b}},\mathcal{F}_{\mathrm{w}}) \cong \mathcal{O}_{\mathfrak{X}_{N}}\Big(\mathbf{D}(\mathrm{w})-\mathbf{D}(\mathrm{b})+\sum_{\rho\in\Sigma(1)}\sum_{\alpha\in Z_{\rho}:\mathrm{b}\in\alpha}\frac{1}{|E_{\rho}|}D_{\rho}\Big).$$

Let $m \in M$ be such that $\phi(e) = \chi^m$ and let $D := \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho}$. By Proposition A.3, χ^m is a global section of the line bundle $\mathcal{O}_{\chi_N}(D)$ if and only if $m \in P_D \cap M$. Using (52), this is equivalent to showing that for every edge e = bw, we have

$$\operatorname{div} \phi(e) + \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \ge 0,$$

where div $\phi(e) = \sum_{\rho \in \Sigma(1)} \langle m, u_{\rho} \rangle D_{\rho}$ as in (48). Let α, β be the zig-zag paths through e, with $\alpha \in Z_{\sigma}, \beta \in Z_{\rho}$. Then by Lemma 3.2, we have

$$\mathbf{D}(\mathbf{w}) - \mathbf{D}(\mathbf{b}) = -\frac{1}{|E_{\sigma}|} D_{\sigma} - \frac{1}{|E_{\rho}|} D_{\rho} - \operatorname{div} \phi(e)$$

This implies

$$\operatorname{div}\phi(e) + \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\tau \in \Sigma(1)} \sum_{\gamma \in Z_{\tau}: b \in \gamma} \frac{1}{|E_{\tau}|} D_{\tau} = \sum_{\substack{\tau \in \Sigma(1)}} \sum_{\substack{\gamma \in Z_{\tau}: b \in \gamma \\ \gamma \neq \alpha, \beta}} \frac{1}{|E_{\tau}|} D_{\tau} \ge 0.$$
(37)

The small polygon N_{bw} is by definition the Newton polygon of Q_{bw} . By Proposition A.3, this is equivalent to saying that Q_{bw} is a global section of a line bundle $\mathcal{O}_{\mathfrak{X}_N}(Y_{\text{bw}})$, where Y_{bw} is the divisor associated to N_{bw} by the correspondence (52). Now that we have shown that K is a global section of $\mathcal{H}om_{\mathfrak{X}_N}(\mathcal{E}, \mathcal{F})$, we can take exterior powers to find which line bundle $\mathcal{O}_{\mathfrak{X}_N}(Y_{\text{bw}})$ the minor Q_{bw} of K is a global section of.

Taking the determinant of the map (36), we see that det \widetilde{K} is a global section of the line bundle

$$\mathcal{H}om_{\mathfrak{X}_{N}}\Big(\bigwedge_{\mathbf{b}\in B}\mathcal{E}_{\mathbf{b}},\bigwedge_{\mathbf{w}\in W}\mathcal{F}_{\mathbf{w}}\Big)\cong\mathcal{O}_{\mathfrak{X}_{N}}\Big(\sum_{\mathbf{w}\in W}\mathbf{D}(\mathbf{w})-\sum_{\mathbf{b}\in B}\Big(\mathbf{D}(\mathbf{b})-\sum_{\rho\in\Sigma(1)}\sum_{\alpha\in Z_{\rho}:b\in\alpha}\frac{1}{|E_{\rho}|}D_{\rho}\Big)\Big).$$
(38)

Lemma 5.2. Let D_N be the divisor associated to N by the correspondence (52) between divisors and polygons. Then one has

$$\sum_{\mathbf{w}\in W} \mathbf{D}(\mathbf{w}) - \sum_{\mathbf{b}\in B} \left(\mathbf{D}(\mathbf{b}) - \sum_{\rho\in\Sigma(1)} \sum_{\alpha\in Z_{\rho}: b\in\alpha} \frac{1}{|E_{\rho}|} D_{\rho} \right) = D_N.$$
(39)

Therefore,

$$\det \widetilde{K} \in H^0(\mathfrak{X}_N, \mathcal{O}_{\mathfrak{X}_N}(D_N)).$$

Proof. Let a_{ρ} be the coefficient of D_{ρ} in D_N . Let (i_1, i_2) be a vertex of P contained in E_{ρ} and let m be the associated extremal dimer cover. We pair up black and white vertices in the sum according to m:

$$\sum_{e=bw\in m} \left(\mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \right).$$

Now we observe that if e is not contained in any zig-zag path in Z_{ρ} , then D_{ρ} does not appear in the summand, and if e is contained in a zig-zag path associated to E_{ρ} , then D_{ρ} appears twice but with opposite signs, modulo contributions from intersections of edges with γ_z, γ_w . Therefore, there is no net contribution to the coefficient of D_{ρ} except for the intersections of edges in m with γ_z, γ_w , which is the same as in

$$-\sum_{e \in \mathbf{m}} \operatorname{div} \phi(e) = -\operatorname{div} z^{i_1} w^{i_2},$$

which is a_{ρ} . Comparing with (38), we see that (39) implies the second statement.

Now we consider the codimension 1 exterior power, where we remove $\{\mathbf{b}, \mathbf{w}\}$. Let \widetilde{Q} be the adjugate matrix of \widetilde{K} . Set

$$Y_{\mathrm{bw}} := D_N - \mathbf{D}(\mathrm{w}) + \mathbf{D}(\mathrm{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho}.$$

Corollary 5.3. $\widetilde{Q}_{\mathrm{bw}} \in H^0(\mathfrak{X}_N, \mathcal{O}_{\mathfrak{X}_N}(Y_{\mathrm{bw}})).$

We therefore arrive at the definition of the small polygon $N_{\rm bw}$ given in Definition 3.3 by the correspondence (52).

5.2 Points at infinity

In this section, we prove that the points at infinity of \mathcal{C} are as described in Section 2.7. We use the notations $U_{\Sigma} \subset \mathbb{C}^{\Sigma(1)}$ and the standard coordinates (z_{ρ}) on $\mathbb{C}^{\Sigma(1)}$ from Appendix A.3. The toric variety X_N is the quotient U_{Σ}/H , where H is the kernel of the map $(\mathbb{C}^{\times})^{\Sigma(1)} \to \mathbb{T}$ sending $(z_{\rho})_{\rho \in \Sigma(1)}$ to $(\prod_{\rho} z_{\rho}^{\langle (1,0), u_{\rho} \rangle}, \prod_{\rho} z_{\rho}^{\langle (0,1), u_{\rho} \rangle})$. There is a canonical map $\pi : U_{\Sigma} \to X_N$ given by sending (z_{ρ}) to $H \cdot (z_{\rho}^{|E_{\rho}|})$ which induces the coarse moduli space map $\pi : \mathfrak{X}_N \to X_N$. The spectral curve \mathcal{C} is cut out by the section P = P(z, w) of $\mathcal{O}_{X_N}(D_N)$. The pullback π^*P defines a section of $\mathcal{O}_{X_N}(D_N)$ which is a G-invariant section of $\mathcal{O}_{U_{\Sigma}}$, so it vanishes on a G-invariant subvariety $\mathcal{C}_{U_{\Sigma}}$. Each point at infinity of \mathcal{C} corresponds to a G-invariant set of points at infinity of $\mathcal{C}_{U_{\Sigma}}$, so we will determine the points at infinity of \mathcal{C} from the points at infinity of $\mathcal{C}_{U_{\Sigma}}$. By Lemma 5.2, $\pi^*P = \det \widetilde{K}$ so the points at infinity of $\mathcal{C}_{U_{\Sigma}}$ are obtained by setting $z_{\rho} = \det \widetilde{K} = 0$ for $\rho \in \Sigma(1)$.

From (37) and Proposition A.3, for e = bw, we get that $\phi(e)$ corresponds to the *G*-invariant section of $\mathcal{O}_{U_{\Sigma}}$ given by

$$\phi(e) = \prod_{\substack{\tau \in \Sigma(1) \\ \gamma \neq \alpha, \beta}} \prod_{\substack{\gamma \in Z_{\tau}: b \in \gamma \\ \gamma \neq \alpha, \beta}} z_{\tau}.$$
(40)

The divisor D_{ρ} in X_N corresponds to $\{z_{\rho} = 0\} \subset U_{\Sigma}$. $\phi(e)$ vanishes on $\{z_{\rho} = 0\}$ precisely when there is a zig-zag path $\alpha \in Z_{\rho}$ such that b is contained in α but w is not contained in α . This implies that when restricted to $\{z_{\rho} = 0\}$, after reordering the black and white vertices, the extended Kastelevn operator \widetilde{K} takes a block-upper-triangular form



where $Z_{\rho} = \{\alpha_1, \ldots, \alpha_n\}, \widetilde{K}\Big|_{\alpha_i}$ is the restriction of \widetilde{K} to the black and white vertices in α_i , and the *'s denote some possibly nonzero blocks whereas any block that has not been indicated is zero. In particular, the nonzero blocks * are only in the last column (in these blocks, b is not in α but w is in α). Note that the zig-zag paths $\alpha_1, \ldots, \alpha_n$ do not share any vertices because of minimiality since otherwise we would have a parallel bigon, so the blocks $\widetilde{K}\Big|_{\alpha_1}, \ldots, \widetilde{K}\Big|_{\alpha_n}$ do not overlap.

If $\alpha \in Z_{\rho}$ is $\mathbf{b}_1 \to \mathbf{w}_1 \to \mathbf{b}_2 \to \cdots \to \mathbf{w}_d \to \mathbf{b}_1$, the determinant of the block \widetilde{K} is

$$\det \widetilde{K}\Big|_{\alpha} = \det \begin{pmatrix} \widetilde{K}_{w_{1}b_{1}} & \widetilde{K}_{w_{2}b_{2}} & \\ \widetilde{K}_{w_{1}b_{2}} & \widetilde{K}_{w_{2}b_{2}} & \\ & \widetilde{K}_{w_{2}b_{3}} & \\ & \ddots & \ddots & \\ & & \widetilde{K}_{w_{d-1}b_{d}} & \widetilde{K}_{w_{d}b_{d}} \end{pmatrix}$$
$$= \prod_{i=1}^{d} \widetilde{K}_{w_{i}b_{i}} - (-1)^{d} \prod_{i=1}^{d} \widetilde{K}_{w_{i-1}b_{i}}$$
$$= -\prod_{i=1}^{d} (wt(\mathbf{b}_{i}w_{i-1})\epsilon(\mathbf{b}_{i}w_{i-1})\phi(\mathbf{b}_{i}w_{i})) \left(\prod_{i=1}^{d} \frac{\phi(\mathbf{b}_{i}w_{i-1})}{\phi(\mathbf{b}_{i}w_{i})} - C_{\alpha} \right)$$

where we have used the definition of the Kasteleyn matrix (see (1), (2). Plugging in (40) and using the fact that α intersects a zig-zag path $\beta \in Z_{\tau} \langle [\alpha], u_{\tau} \rangle$ times, we get

$$\prod_{i=1}^{d} \frac{\phi(\mathbf{b}_{i}\mathbf{w}_{i-1})}{\phi(\mathbf{b}_{i}\mathbf{w}_{i})} = \prod_{\tau \in \Sigma(1)} z_{\tau}^{-|E_{\tau}|\langle [\alpha], u_{\tau} \rangle}.$$

Therefore, the points at infinity of $C_{U_{\Sigma}}$ are given by setting $z_{\rho} = 0$ and $\prod_{\tau \in \Sigma(1)} z_{\tau}^{-|E_{\tau}|\langle [\alpha], u_{\tau} \rangle} = C_{\alpha}$. The point at infinity of C is the point obtained by applying π to any of these points. From the definition of π , we get that $\prod_{\tau \in \Sigma(1)} z_{\tau}^{-|E_{\tau}|\langle [\alpha], u_{\tau} \rangle} = \pi^* \chi^{-[\alpha]}$, so the point at infinity of C is given by

$$\chi^{-[\alpha]} = C_{\alpha}.\tag{41}$$

6 Behaviour of the Laurent polynomial $Q_{bw}(z, w)$ at infinity

We proved in Corollary 5.3 that the Laurent polynomial $Q_{\rm bw}(z,w)$ lies in the finite dimensional vector space $H^0(\mathcal{X}_N, \mathcal{O}_{\mathcal{X}_N}(Y_{\rm bw}))$. We need some additional constraints on $Q_{\rm bw}(z,w)$ to determine it. Corollary 6.3 provides g linear equations for the coefficients of $Q_{\rm bw}(z,w)$ coming from the vanishing of $Q_{\rm bw}(z,w)$ at the g points of the divisor $S_{\rm w}$. We obtain additional equations from the behaviour of $Q_{\rm bw}(z,w)$ at the points at infinity of the spectral curve, which we study in this section.

Recall that X_N is the toric surface associated to N compactifying T. The restriction of the Kasteleyn operator to the open spectral curve \mathcal{C}° is a map of trivial sheaves:

$$K|_{\mathcal{C}^{\circ}} : \bigoplus_{\mathbf{b} \in B} \mathcal{O}_{\mathcal{C}^{\circ}} \longrightarrow \bigoplus_{w \in W} \mathcal{O}_{\mathcal{C}^{\circ}}.$$

Recall the correspondence between divisors D and invertible sheaves with rational sections (\mathcal{L}, s) : given a divisor D, the corresponding invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(D)$ is defined on an open U by

$$H^{0}(U, \mathcal{O}_{\mathcal{C}}(D)) := \{ t \in K(\mathcal{C})^{\times} : (\operatorname{div} t + D) \big|_{U} \ge 0 \} \cup \{ 0 \},\$$

with the obvious restriction maps, where $K(\mathcal{C})^{\times}$ denotes the nonzero rational functions on \mathcal{C} . The rational section s corresponds to the rational function 1. On the other hand, given (\mathcal{L}, s) , we obtain

D as the divisor div s. Moreover, there is a correspondence between rational functions t on C and rational sections \overline{t} of \mathcal{L} given by $t \mapsto \overline{t} := st$. In particular,

$$\operatorname{div} \overline{t} = \operatorname{div} t + \operatorname{div} s = \operatorname{div} t + D, \tag{42}$$

and so \overline{t} is regular if and only if div $t + D \ge 0$.

A similar proof to Proposition 5.1 shows that the Kasteleyn matrix K, which is a matrix of rational functions on \mathcal{C} , defines a regular map \overline{K} of locally free sheaves on \mathcal{C} extending $K|_{\mathcal{C}^{\circ}}$, providing an exact sequence

$$0 \to \mathcal{M} \to \bigoplus_{\mathbf{b} \in B} \mathcal{O}_{\mathcal{C}} \left(\mathbf{d}(\mathbf{b}) - \sum_{\alpha \in Z: \mathbf{b} \in \alpha} \nu(\alpha) \right) \xrightarrow{\overline{K}} \bigoplus_{\mathbf{w} \in W} \mathcal{O}_{\mathcal{C}} (\mathbf{d}(\mathbf{w})) \to \mathcal{L} \to 0,$$
(43)

where \mathcal{M} and \mathcal{L} are the kernel and cokernel of the map \overline{K} respectively. When we say \overline{K} is regular, we mean that each entry \overline{K}_{wb} is a regular section of the corresponding $\mathcal{H}om$ line bundle

$$\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}\Big(\mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathbf{b}) - \sum_{\alpha \in Z: \mathbf{b} \in \alpha} \nu(\alpha)), \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathbf{w}))\Big).$$

For generic dimer weights, Σ is smooth, and \mathcal{M} and \mathcal{L} are line bundles (so \overline{Q} has rank 1). Let \overline{s}_{b} and \overline{s}_{w} be sections of

$$\mathcal{M}^{\vee} \otimes \mathcal{O}_{\mathcal{C}} \Big(\mathbf{d}(\mathbf{b}) - \sum_{\alpha \in Z: \mathbf{b} \in \alpha} \nu(\alpha) \Big) \text{ and } \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}} (\mathbf{d}(\mathbf{w}))^{\vee}$$

respectively, given by the b-entry of the kernel map and w-entry of the cokernel map respectively. Since \overline{Q} has rank 1, we have $\overline{Q}_{bw} = \overline{s}_b \overline{s}_w$. Denote by S_b and S_w the effective divisors on the open spectral curve \mathcal{C}° given by vanishing of the b-row and w-column of Q respectively, or equivalently, the vanishing of \overline{s}_b and \overline{s}_w respectively.

Lemma 6.1.

$$\begin{split} \operatorname{div}_{\mathcal{C}} \overline{s}_{\mathrm{b}} &= S_{\mathrm{b}} + \sum_{\alpha \in Z: \mathrm{b} \notin \alpha} \nu(\alpha) \\ \operatorname{div}_{\mathcal{C}} \overline{s}_{\mathrm{w}} &= S_{\mathrm{w}}, \end{split}$$

Proof. By the definition, $(\operatorname{div}_{\mathcal{C}} \overline{s}_{\mathrm{b}})|_{\mathcal{C}^{\circ}} = S_{\mathrm{b}}$ and $(\operatorname{div}_{\mathcal{C}} \overline{s}_{\mathrm{w}})|_{\mathcal{C}^{\circ}} = S_{\mathrm{w}}$, so it only remains to find their orders of vanishing at infinity.

Let $U \subset \mathcal{C}$ be a neighbourhood of $\nu(\alpha)$ that does not contain any other point at infinity. Let u be a local parameter in U that vanishes to order 1 at $\nu(\alpha)$ and nowhere else. When restricted to U, each of the line bundles in the source and target of \overline{K} in (43) is of the form $\mathcal{O}_U(k\nu(\alpha))$ for some $k \in \mathbb{Z}$. We trivialize $\mathcal{O}_U(k\nu(\alpha))$ as follows:

$$\mathcal{O}_U(k\nu(\alpha)) \xrightarrow{\cong} \mathcal{O}_U$$
$$f \mapsto u^k f$$

Let us order the black and white vertices so that the vertices on α come first. Then in U, we have

$$\overline{K} = \begin{pmatrix} K_1 & B\\ uA & K_2 \end{pmatrix} + O(u),$$

where K_1, K_2 are the restrictions of \overline{K} to α and $\Gamma - \alpha$ respectively. Since corank $\overline{K} = 1$ and since we know corank $K_1 > 0$ from the computation of the determinant in Section 5.2, we have corank $K_1 = 1$ and that K_2 is invertible. Let $v \in \ker K_1$. Then,

$$\ker \overline{K} = (v, -uK_2^{-1}Av) + O(u).$$

If any entry in v or $K_2^{-1}Av$ is 0, then it means that some \overline{s}_b is identically 0. Let \overline{Q} denote the adjugate matrix of \overline{K} . Since \overline{Q} has rank 1, we have $\overline{Q}_{bw} = \overline{s}_b \overline{s}_w = 0$, so we will have $\overline{Q}_{bw} = 0$ for all $w \in W$. On the other hand, Q_{bw} is the signed partition function for dimer covers of $\Gamma \setminus \{b, w\}$, so if we choose w such that bw is an edge of Γ used in a dimer cover, then $Q_{bw} \neq 0$ for generic dimer weights, a contradiction. Therefore, the entries of ker \overline{K} are nonzero when $u \neq 0$, so \overline{s}_b has a simple zero at $\nu(\alpha)$ for all $b \notin \alpha$ and has no zeroes or poles for $b \in \alpha$.

Similarly, let $v' \in \ker K_1^*$. We have

$$\ker \overline{K}^* = (v', -(K_2^*)^{-1}Bv') + O(u)$$

For generic dimer weights, none of the entries of $(K_2^*)^{-1}Bv'$ can vanish, so \overline{s}_w has no zeroes or poles at $\nu(\alpha)$.

Corollary 6.2. div_C $Q_{bw} = S_b + S_w - D_N |_{\mathcal{C}} + \mathbf{d}(w) - \mathbf{d}(b) + \sum_{\alpha \in Z} \nu(\alpha).$

Proof. Let \overline{Q} denote the adjugate matrix of \overline{K} . Since \overline{Q} has rank 1, we have $\overline{Q}_{bw} = \overline{s}_b \overline{s}_w$, so that

$$\operatorname{div}_{\mathcal{C}} \overline{Q}_{\mathrm{bw}} = S_{\mathrm{b}} + S_{\mathrm{w}} + \sum_{\alpha \in Z: \mathrm{b} \notin \alpha} \nu(\alpha).$$

A computation similar to Corollary 5.3 shows that

$$\overline{Q}_{\mathrm{bw}} \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D_N\big|_{\mathcal{C}} + \mathbf{d}(\mathrm{w}) - \mathbf{d}(\mathrm{b}) + \sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha))).$$

 $Q_{\rm bw}$ is the rational function corresponding to the rational section $\overline{Q}_{\rm bw}$. Therefore, using (42), we have

$$\operatorname{div}_{\mathcal{C}} Q_{\mathrm{bw}} = \operatorname{div}_{\mathcal{C}} \overline{Q}_{\mathrm{bw}} - D_N \big|_{\mathcal{C}} + \mathbf{d}(\mathrm{w}) - \mathbf{d}(\mathrm{b}) + \sum_{\alpha \in Z: \mathrm{b} \in \alpha} \nu(\alpha)$$
$$= S_{\mathrm{b}} + S_{\mathrm{w}} - D_N \big|_{\mathcal{C}} + \mathbf{d}(\mathrm{w}) - \mathbf{d}(\mathrm{b}) + \sum_{\alpha \in Z} \nu(\alpha).$$

Corollary 6.3. We have for all $b \in B, w \in W$, deg $S_b = \deg S_w = g$, where g is the genus of C.

Proof. Let ω_* denote the canonical divisor of *. We have $\omega_{X_N} = -\sum_{\rho \in \Sigma(1)} D_{\rho}$ ([CLS11, Theorem 8.2.3]). By the adjunction formula, we get $\omega_{\mathcal{C}} = D_N |_{\mathcal{C}} - \sum_{\alpha \in Z} \nu(\alpha)$. Since Q_{bw} is a rational function on \mathcal{C} , we have deg $\operatorname{div}_{\mathcal{C}} Q_{\text{bw}} = 0$. Since $\operatorname{deg}(\mathbf{d}(w) - \mathbf{d}(b)) = -2$ and $\operatorname{deg} \omega_{\mathcal{C}} = 2g - 2$, we get $\operatorname{deg}(S_b + S_w) = 2g$. By symmetry under interchanging B and W, we get $\operatorname{deg} S_b = \operatorname{deg} S_w = g$.

Recall that the number g is also equal to the number of interior lattice points in N for generic $C \in |D_N|$.

Proposition 6.4. The line bundle \mathcal{L} is isomorphic to $\mathcal{O}_{\mathcal{C}}(S_{w} + \mathbf{d}(w))$ for any $w \in W$. It has degree g - 1.

Proof. By Lemma 6.1, \overline{s}_{w} is a section of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w))^{\vee}$ with divisor S_{w} . Therefore, we must have

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(\mathbf{w}))^{\vee} \cong \mathcal{O}_{\mathcal{C}}(S_{\mathbf{w}}),$$

which implies $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(S_{w} + \mathbf{d}(w))$. Since deg $S_{w} = g$ and deg $\mathbf{d}(w) = -1$, we get deg $\mathcal{L} = g - 1$. \Box

7 Equations for the Laurent polynomial $Q_{\rm bw}$

Since Q_{bw} has Newton polygon N_{bw} , we have

$$Q_{\mathrm{b}\mathbf{w}} = \sum_{m \in N_{\mathrm{b}\mathbf{w}} \cap \mathrm{M}} a_m \chi^m,$$

for some $a_m \in \mathbb{C}$. We know that $Q_{\mathbf{bw}}$ vanishes on $S_{\mathbf{w}}$, which gives g linear equations among the $(a_m)_{m \in N_{\mathbf{bw}} \cap \mathbf{M}}$. However these g linear equations are not usually sufficient to determine $Q_{\mathbf{bw}}$, so we need to find some additional equations. These additional equations will come from the vanishing of $Q_{\mathbf{bw}}$ at the points at infinity.

7.1 Additional linear equations for Q_{bw}

The fact that the Newton polygon of $Q_{\mathbf{bw}}$ is the small polygon $N_{\mathbf{bw}}$ imposes certain inequalities on the order of vanishing of $Q_{\mathbf{bw}}$ at points at infinity of C. Corollary 6.2 imposes additional constraints that are linear equations in the coefficients of $Q_{\mathbf{bw}}$. Inverting this linear system gives $(a_m)_{m \in N_{\mathbf{bw}} \cap \mathbf{M}}$ and therefore $Q_{\mathbf{bw}}$.

We now give the precise statement. For a \mathbb{Q} -divisor $D = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$, we define a (\mathbb{Z} -) divisor $\lfloor D \rfloor := \sum_{\rho \in \Sigma(1)} \lfloor b_{\rho} \rfloor D_{\rho}$, where $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$. It is the pushforward of D by the canonical projection $\mathfrak{X}_N \to X_N$.

Proposition 7.1. The extra linear equations for $(a_m)_{m \in N_{bw} \cap M}$ from vanishing of Q_{bw} at points at infinity correspond to the points in

$$-D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} \nu(\alpha) + \lfloor Y_{\mathbf{b}\mathbf{w}} \rfloor \Big|_{\mathcal{C}}.$$
(44)

Proof. A generic Laurent polynomial F of the form $\sum_{m \in N_{bw} \cap M} a_m \chi^m$ has order of vanishing

$$\operatorname{div}_{\mathcal{C}} F\big|_{\mathcal{C}} \ge -\lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor\big|_{\mathcal{C}}$$

at the points at infinity of C. From Corollary 6.2, we have that $\operatorname{div}_{\mathcal{C}} Q_{\mathrm{b}\mathbf{w}} = S_{\mathrm{b}} + S_{\mathbf{w}} - D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathrm{b}) + \sum_{\alpha \in \mathbb{Z}} \nu(\alpha)$. The discrepancy provides the extra equations.

Now we describe these extra linear equations explicitly. Suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes a linear equation. We extend $[\alpha]$ to a basis $([\alpha] = x_1, x_2)$ of M such that $\langle x_2, u_{\rho} \rangle = 1$, so that for any $m \in \mathcal{M}$, we can write

$$\chi^m = x_1^{b_m} x_2^{c_m}, \quad b_m, c_m \in \mathbb{Z}$$

Let N_{bw}^{ρ} be the set of lattice points in N_{bw} closest to the edge E_{ρ} of N i.e., the set of points in N_{bw} that minimize the functional $\langle *, u_{\rho} \rangle$.

Proposition 7.2. Suppose $Q_{bw} = \sum_{m \in N_{bw} \cap M} a_m \chi^m$ and suppose $\alpha \in Z_{\rho}$ is a zig-zag path that contributes a linear equation. Then, the linear equation given by α is:

$$\sum_{n \in N_{\rm bw}^{\rho} \cap \mathcal{M}} a_m C_{\alpha}^{-b_m} = 0$$

Proof. The affine open variety in X_N corresponding to the cone ρ is

$$U_{\rho} = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, x_2] \cong \mathbb{C}^{\times} \times \mathbb{C},$$

and $D_{\rho} \cap U_{\rho}$ is defined by $x_2 = 0$.

A generic curve \mathcal{C} meets D_{ρ} transversely at $\nu(\alpha)$, and therefore we may take x_2 as a uniformizer of the local ring $\mathcal{O}_{\mathcal{C},\nu(\alpha)}$ at $\nu(\alpha)$. For each $m \in N_{\mathrm{bw}}^{\rho} \cap \mathcal{M}$, we have

$$\chi^m = x_1^{b_m} x_2^p, \quad b_\gamma, p \in \mathbb{Z},$$

where p is the same for all of them, and is the coefficient of $\nu(\alpha)$ in $-[E_{\mathbf{b}w}]|_{\mathcal{C}}$. Then using $x_1^{-1} = C_{\alpha}$ at $\nu(\alpha)$, we have

$$Q_{\mathrm{b}\mathbf{w}} = \left(\sum_{m \in N_{\mathrm{b}\mathbf{w}}^{\rho} \cap \mathrm{M}} a_m C_{\alpha}^{-b_m}\right) x_2^p + O(x_2^{p+1}).$$

$$\tag{45}$$

Since α contributes a linear equation, (45) must vanish at order x_2^p , so $\sum_{m \in N_{h,m}^{\rho} \cap M} a_m C_{\alpha}^{-b_m} = 0$. \square

7.2 The system of linear equations \mathbb{V}_{bw}

Recall from Section 3 the system of linear equations \mathbb{V}_{bw} . These are linear equations in the variables $(a_m)_{m \in N_{bw} \cap M}$. Recall also that the matrix \mathbb{V}_{bw} is defined such that these equations are given by

$$\mathbb{V}_{\mathbf{b}\mathbf{w}}(a_m) = 0.$$

It is not necessarily a square matrix. However, we have:

Proposition 7.3. For generic spectral data, Q_{bw} is the unique solution of the linear system of equations \mathbb{V}_{bw} modulo scaling.

- **Remark 7.4.** 1. While the definition of \mathbb{V}_{bw} makes sense for all $w \in W$, Proposition 7.3 only holds when $w = \mathbf{w}$ since $(p_i, q_i)_{i=1}^g$ depends on \mathbf{w} .
 - 2. For generic spectral data, the equations (23) are linearly independent, but the equations (25) may not be.

The rest of this section is devoted to the proof of Proposition 7.3. Consider following the exact sequence on X_N , obtained by tensoring the closed embedding exact sequence of $i : \mathcal{C} \hookrightarrow X_N$ by $\mathcal{O}_{X_N}(\lfloor Y_{\mathbf{bw}} \rfloor)$.

$$0 \to \mathcal{O}_{X_N}(\lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor - D_N) \to \mathcal{O}_{X_N}(\lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor) \to i_*\mathcal{O}_{\mathcal{C}}(\lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor |_{\mathcal{C}}) \to 0.$$

The following is a portion of the long exact sequence of cohomology.

$$0 \to H^0(X_N, \lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor - D_N) \to H^0(X_N, \lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor) \to H^0(\mathcal{C}, \lfloor Y_{\mathrm{b}\mathbf{w}} \rfloor \big|_{\mathcal{C}}).$$
(46)

We need the following technical lemma.

Lemma 7.5. The restriction map $H^0(X_N, \lfloor Y_{\mathrm{bw}} \rfloor) \to H^0(\mathcal{C}, \lfloor Y_{\mathrm{bw}} \rfloor)$ is injective.

Proof. If $\chi^m \in H^0(X_N, \lfloor Y_{\mathbf{bw}} \rfloor - D_N)$, then div $\chi^m + \lfloor Y_{\mathbf{bw}} \rfloor - D_N \ge 0$. This implies that

$$\operatorname{div} \chi^{m} + Y_{\mathrm{b}\mathbf{w}} - D_{N} = \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}} \langle m, u_{\rho} \rangle \frac{D_{\rho}}{|E_{\rho}|} - \mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathrm{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathrm{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \ge 0.$$

$$(47)$$

Let γ be a cycle in \mathbb{T} with homology class m. The total number of signed intersections of γ with all zig-zag paths is zero. This number is the sum of the coefficients of $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}} \langle m, u_{\rho} \rangle \frac{D_{\rho}}{|E_{\rho}|}$. Let w' be any white vertex adjacent to b. Then we have

$$-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} = (\mathbf{D}(\mathbf{w}') - \mathbf{D}(\mathbf{w})) - \sum_{\substack{\rho \in \Sigma(1)}} \sum_{\substack{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha \\ \mathbf{b} \mathbf{w}' \notin \alpha}} \frac{1}{|E_{\rho}|} D_{\rho}.$$

The sum of the coefficients of $\mathbf{D}(\mathbf{w}') - \mathbf{D}(\mathbf{w})$ is the signed number of intersections with zigzag paths of any path in R from \mathbf{w} to \mathbf{w}' , which is also 0. Since the coefficients of the last term $-\sum_{\substack{\rho \in \Sigma(1) \\ \mathbf{b}\mathbf{w}' \notin \alpha}} \sum_{\substack{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha \\ \mathbf{b}\mathbf{w}' \notin \alpha}} \frac{1}{|E_{\rho}|} D_{\rho}$ are strictly negative, the sum in (47) cannot be non-negative. Therefore, $H^0(X_N, [Y_{\mathbf{b}\mathbf{w}}] - D_N) = 0$, which by (46) means that the map $H^0(X_N, [Y_{\mathbf{b}\mathbf{w}}]) \rightarrow$ $H^0(\mathcal{C}, [Y_{\mathbf{b}\mathbf{w}}]|_{\mathcal{C}})$ is injective.

- Proof of Proposition 7.3. 1. Existence: By Theorem 7.3 of [GK13], the map $\kappa_{\Gamma,\mathbf{w}}$ is dominant. So a generic spectral data is in the image of $\kappa_{\Gamma,\mathbf{w}}$. For such a spectral data, $Q_{\mathbf{bw}}$ satisfies:
 - (a) The system of equations (23) because, by definition of the spectral transform, Q_{bw} vanishes at the points of the divisor $S = \sum_{i=1}^{g} (p_i, q_i)$.
 - (b) The equations (25) by Proposition 7.2.
 - 2. Uniqueness: Suppose V_{bw} is a solution of \mathbb{V}_{bw} . Since V_{bw} has Newton polygon N_{bw} , we have $\operatorname{div}_{\mathcal{C}} F|_{\mathcal{C}} \geq -\lfloor Y_{bw} \rfloor|_{\mathcal{C}}$ as in the proof of Proposition 7.1. The additional equations in Proposition 7.1 then imply that

$$\operatorname{div}_{\mathcal{C}} V_{\mathrm{b}\mathbf{w}} \ge S + D,$$

where $D := -D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in \mathbb{Z}} \nu(\alpha)$ satisfies deg D = -2g. Therefore, $V_{\mathbf{b}\mathbf{w}}|_{\mathcal{C}}$ can be identified with a section of $\mathcal{O}_{\mathcal{C}}(-D)$ vanishing at the points of S. Let $\omega_{\mathcal{C}}$ denote the canonical divisor of \mathcal{C} as in Section 6. By the Riemann-Roch theorem,

$$h^{0}(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D) - h^{1}(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)) = \deg(-D) - g + 1 = g + 1.$$

By Serre duality, $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)) = h^0(\mathcal{C}, \omega_{\mathcal{C}}(D))$, which equals 0 since $\omega_{\mathcal{C}}(D)$ has negative degree -2. For generic S that avoids the base locus of $\mathcal{O}_{\mathcal{C}}(-D)$, the requirement that the section of $\mathcal{O}_{\mathcal{C}}(-D)$ corresponding to V_{bw} vanishes at each of the g points of S imposes g independent conditions, and therefore determines $V_{\mathrm{bw}}|_{\mathcal{C}}$ uniquely up to multiplication by a nonzero complex number. By Lemma 7.5, V_{bw} is unique up to multiplication by a nonzero complex number.

Remark 7.6. It is easy to see using Riemann-Roch that the number of equations in \mathbb{V}_{bw} is equal to $h^0(\mathcal{C}, \lfloor Y_{bw} \rfloor \mid_{\mathcal{C}}) - 1$. On the other hand, the number of variables is $h^0(X_N, \lfloor Y_{bw} \rfloor)$. However, the map in Lemma 7.5 is not necessarily an isomorphism (there may be sections on the curve that are not restrictions of sections on the surface), so we only have the inequality

equations in $\mathbb{V}_{bw} \ge \#$ variables -1.

A Toric geometry

In A.1 and A.2, we give a brief background on toric varieties; further details can be found in the books [Ful93] and [CLS11].

A.1 Toric varieties

A toric variety X over \mathbb{C} is an algebraic variety containing the complex algebraic torus $T \cong (\mathbb{C}^{\times})^n$ as a Zariski open subset, such that the action of T on itself extends to an action of T on X.

Let M be a lattice, and let $M^{\vee} := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ denote the dual lattice. Let $T := M^{\vee} \otimes \mathbb{C}^{\times} = \operatorname{Hom}(M, \mathbb{C}^{\times})$ be the complex algebraic torus with the lattice of characters M. We denote by $\chi^m : T \to \mathbb{C}^{\times}$ the character associated to $m \in M$. Let $\langle *, * \rangle$ be the pairing between M and M^{\vee} . In our case $M = H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$, so $M^{\vee} = H^1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ and $T = H^1(\mathbb{T}, \mathbb{C}^{\times}) \cong (\mathbb{C}^{\times})^2$. We have $\chi^{(i,j)}(z, w) = z^i w^j$.

A fan Σ is a collection of cones in the real vector space $M_{\mathbb{R}}^{\vee} := M^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$, which is just the Lie algebra of the real torus $T(\mathbb{R})$, such that

- 1. Each face of a cone $\sigma \in \Sigma$ is also in Σ .
- 2. The intersection of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of each of them.

Each cone $\sigma \in \Sigma$ gives rise to an affine toric variety

$$\mathbf{U}_{\sigma} = \operatorname{Spec} \mathbb{C}[S_{\sigma}],$$

where $S_{\sigma} = \sigma^{\vee} \cap M$ is a semigroup, σ^{\vee} is the cone dual to σ , and $\mathbb{C}[S_{\sigma}]$ is its semigroup algebra:

$$\mathbb{C}[S_{\sigma}] = \left\{ \sum_{m \in S_{\sigma}} c_m \chi^m : c_m \in \mathbb{C}, c_m = 0 \text{ for all but finitely many } m \in S_{\sigma} \right\}.$$

If $\tau \subset \sigma$, then U_{τ} is an open subset of U_{σ} . Gluing the affine toric varieties $U_{\sigma_1}, U_{\sigma_2}$ along $U_{\sigma_1 \cap \sigma_2}$ for all cones $\sigma_1, \sigma_2 \in \Sigma$, we get the toric variety X_{Σ} associated to Σ .

In particular, if $\sigma = \{0\}$, then $S_{\sigma} = M$, so $\mathbb{C}[S_{\sigma}] = \mathbb{C}[M]$ and $U_{\sigma} = T$. So X_{Σ} contains T. We define the action $T \times U_{\sigma} \longrightarrow U_{\sigma}$ via the dual map of the algebras of functions:

e define the action
$$1 \times 0_{\sigma} \longrightarrow 0_{\sigma}$$
 via the dual map of the algebras of functions

$$\mathbb{C}[S_{\sigma}] \longrightarrow \mathbb{C}[\mathbf{M}] \otimes \mathbb{C}[S_{\sigma}],$$
$$\chi^m \longmapsto \chi^m \otimes \chi^m.$$

When $\sigma = \{0\}$, this is the action of T on itself. The action of T on U_{σ} is compatible with the gluing, and therefore gives an action of T on X_{Σ} .

We denote by $\Sigma(r)$ the set of r-dimensional cones of Σ . There is an inclusion-reversing bijection between T-orbit closures in X_{Σ} and cones in Σ . Under this bijection, each ray $\rho \in \Sigma(1)$ corresponds to a T-invariant divisor D_{ρ} . Let u_{ρ} be the primitive vector generating ρ . Then, the (Weil) divisor of the character χ^m is

$$\operatorname{div} \chi^{m} = \sum_{\rho \in \Sigma(1)} \langle m, u_{\rho} \rangle D_{\rho}.$$
(48)

The following fundamental exact sequence computes the class group of Weil divisors of X_{Σ} :

$$0 \to \mathcal{M} \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_{\Sigma}) \to 0,$$

$$m \mapsto (\langle m, u_{\rho} \rangle)_{\rho \in \Sigma(1)}.$$

$$(49)$$

A.2 Polygons and projective toric surfaces

Given a convex integral polygon N in the plane $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, we construct the normal fan Σ of N as follows:

- 1. $\Sigma(0) = \{0\}.$
- 2. For each edge E_{ρ} of N, let $u_{\rho} \in \mathbf{M}^{\vee}$ be the primitive inward normal vector to E_{ρ} , providing an element of $\Sigma(1)$ given by the ray spanned by u_{ρ} .
- 3. For each vertex v of N, we get an element of $\Sigma(2)$ by taking the convex hull of the two rays in $\Sigma(1)$ associated to the two edges incident to v in N.

The normal fan Σ gives rise to a toric surface denoted below by X_N . The orbit-cone correspondence assigns to each edge E_{ρ} of N a divisor $D_{\rho} \cong \mathbb{P}^1$. These divisors intersect according to the combinatorics of N. Their union is the *divisor at infinity* $X_N - T$.

In fact the polygon N determines a pair (X_N, D_N) , where D_N is an ample divisor at infinity:

$$D_N := \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where a_{ρ} is such that the edge E_{ρ} of N is contained in the line $\{m \in \mathcal{M} \otimes \mathbb{R} : \langle m, u_{\rho} \rangle = -a_{\rho}\}$. The linear system of hyperplane sections $|D_N|$ has the following properties:

- 1. $H^0(X_N, \mathcal{O}_{X_N}(D_N)) \cong \bigoplus_{m \in N \cap M} \mathbb{C} \cdot \chi^m$.
- 2. The genus of a generic curve C in $|D_N|$ is the number of interior lattice points of N.
- 3. Curves in $|D_N|$ intersect the divisor D_{ρ} with multiplicity $|E_{\rho}|$ (the number of primitive vectors in E_{ρ}).

A.3 Toric stacks

We follow [BH09, Section 2]. Given a convex integral polygon $N \subset M_{\mathbb{R}}$, we define a *stacky fan* Σ as the following data:

- 1. The normal fan Σ of N, defined above.
- 2. For each ray $\rho \in \Sigma(1)$, the vector $|E_{\rho}|u_{\rho}$ generating the ray ρ .

We define a fan $\widetilde{\Sigma} \subset \mathbb{R}^{\Sigma(1)}$ as follows: for $\sigma \in \Sigma$, we define $\widetilde{\sigma} \in \widetilde{\Sigma}$ by

$$\widetilde{\sigma} = \operatorname{cone}(e_{\rho} : \rho \in \sigma(1)) \subset \mathbb{R}^{\Sigma(1)}$$

where $\{e_{\rho}\}$ is the standard basis in ρ in $\mathbb{R}^{\Sigma(1)}$, and $\sigma(1)$ denotes the rays of Σ incident to σ . Then $\widetilde{\Sigma}$ is the fan generated by the cones $\widetilde{\sigma}$ and their faces.

Let U_{Σ} be the toric variety of the fan $\tilde{\Sigma}$. It is of the form $\mathbb{C}^{\Sigma(1)}$ – (closed codimension 2 subset). Consider the following map, modifying the map (49) for polygons N with a non-primitive side:

$$\beta: \mathbf{M} \to \mathbb{Z}^{\Sigma(1)}$$
$$m \mapsto (|E_{\rho}|\langle m, u_{\rho} \rangle)_{\rho}.$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(*, \mathbb{C}^{\times})$, we get a surjective map $(\mathbb{C}^{\times})^{\Sigma(1)} \to \mathbb{T}$. Denote by G its kernel. So there is an exact sequence

$$1 \to G \to (\mathbb{C}^{\times})^{\Sigma(1)} \to \mathcal{T} \to 1.$$
(50)

So G is a subgroup of the torus $(\mathbb{C}^{\times})^{\Sigma(1)}$ of the toric variety U_{Σ} . Therefore, G acts on U_{Σ} . Explicitly, $\lambda = (\lambda_{\rho}) \in (\mathbb{C}^{\times})^{\Sigma(1)}$ is in G if and only if

$$\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{|E_{\rho}|\langle m, u_{\rho} \rangle} = 1$$
(51)

for all $m \in M$. Let $z = (z_{\rho}) \in \mathbb{C}^{\Sigma(1)}$ denote the standard coordinates on $\mathbb{C}^{\Sigma(1)}$. The action of G on U_{Σ} is $\lambda \cdot z = (\lambda_{\rho} z_{\rho})$.

Definition A.1. The toric stack \mathfrak{X}_N is the smooth Deligne-Mumford stack $[U_{\Sigma}/G]$.

A.4 Example: a stacky \mathbb{P}^2 .

Consider the polygon N given by the convex-hull of $\{(0,0), (2,0), (0,2)\}$. The rays of its normal fan Σ are generated by $u_1 = (1,0), u_2 = (0,1), u_3 = (-1,-1)$ with $|E_1| = |E_2| = |E_3| = 2$. The fan $\widetilde{\Sigma} \subset \mathbb{R}^3$ is generated by the cones

$$\widetilde{\sigma}_1 = \operatorname{cone}(e_2, e_3), \quad \widetilde{\sigma}_2 = \operatorname{cone}(e_1, e_3), \quad \widetilde{\sigma}_3 = \operatorname{cone}(e_1, e_2),$$

and their faces, where $\{e_i\}$ is the standard basis of \mathbb{R}^3 . These cones define affine varieties

$$U_1 = \operatorname{Spec} \mathbb{C}[X_1^{\pm 1}, X_2, X_3], \quad U_2 = \operatorname{Spec} \mathbb{C}[X_1, X_2^{\pm 1}, X_3], \quad U_3 = \operatorname{Spec} \mathbb{C}[X_1, X_2, X_3^{\pm 1}],$$

respectively. The face $\tilde{\sigma}_{12} := \tilde{\sigma}_1 \cap \tilde{\sigma}_2 = \operatorname{cone}(e_3)$ defines the affine variety $U_{12} = \operatorname{Spec} \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, X_3]$, identified with $U_1 \cap U_2$. Similarly, we define U_{23} and U_{13} . Gluing U_i and U_j along the U_{ij} for all i, j, we see that the toric variety U_{Σ} of $\tilde{\Sigma}$ is $\mathbb{C}^3 - 0$. The map $M \to \mathbb{Z}^{\Sigma(1)}$ is

$$\mathbb{Z}^2 \to \mathbb{Z}^3$$
$$(1,0) \mapsto (2,0,-2)$$
$$(0,1) \mapsto (0,2,-2)$$

The group G is the kernel of

$$(\mathbb{C}^{\times})^3 \to (\mathbb{C}^{\times})^2$$
$$(t_1, t_2, t_3) \mapsto \left(\left(\frac{t_1}{t_3} \right)^2, \left(\frac{t_2}{t_3} \right)^2 \right)$$

Thus, $G = \{(\pm \lambda, \pm \lambda, \lambda) : \lambda \in \mathbb{C}^{\times}\}$ and it acts on $\mathbb{C}^3 - 0$ by multiplication. The quotient $[\mathbb{C}^3 - 0/G]$ is a stacky \mathbb{P}^2 .

A.5 Line bundles and divisors on toric stacks

A line bundle on the quotient stack $\mathfrak{X}_N = [U_{\Sigma}/G]$ is the same thing as a *G*-equivariant line bundle on U_{Σ} . The Picard group of U_{Σ} is trivial, so line bundles on \mathfrak{X}_N correspond to the various *G*linearizations of $\mathcal{O}_{U_{\Sigma}}$.

Proposition A.2 (Borisov and Hua, 2009 [BH09, Proposition 3.3]). There is an isomorphism, describing the Picard group of X_N via divisors D_{ρ} :

$$\mathbb{Z}^{\Sigma(1)}/\beta^* \mathbf{M} \cong \operatorname{Pic} \mathfrak{X}_N,$$
$$(b_{\rho})_{\rho} \mapsto \mathcal{O}_{\mathfrak{X}_N} \Big(\sum_{\rho \in \Sigma(1)} \frac{b_{\rho}}{|E_{\rho}|} D_{\rho} \Big).$$

The line bundle $\mathcal{O}_{\mathfrak{X}_N}\left(\sum_{\rho\in\Sigma(1)}\frac{b_{\rho}}{|E_{\rho}|}D_{\rho}\right)$ is the trivial line bundle $\mathcal{O}_{U_{\Sigma}} = U_{\Sigma}\times\mathbb{C}$ with the *G*-linearization

$$G \times (U_{\Sigma} \times \mathbb{C}) \to U_{\Sigma} \times \mathbb{C}$$
$$\lambda \cdot (z, t) \mapsto \left(\lambda \cdot z, t \prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}}\right).$$

Let $D = \sum_{\rho \in \Sigma(1)} \frac{b_{\rho}}{|E_{\rho}|} D_{\rho}$ be a divisor at infinity on \mathfrak{X}_N . We assign to D a polygon P_D in $\mathcal{M}_{\mathbb{R}}$ defined by the intersection of the half planes provided by the coefficients of D:

$$P_D := \bigcap_{\rho \in \Sigma(1)} \left\{ m \in \mathcal{M}_{\mathbb{R}} : \langle m, u_\rho \rangle \ge -\frac{b_\rho}{|E_\rho|} \right\}.$$
(52)

A global section of a line bundle on \mathcal{X}_N is the same thing as a *G*-invariant global section of $\mathcal{O}_{U_{\Sigma}}$. As in the case of toric varieties, global sections of toric line bundles are identified with integral points in the associated polygons:

Proposition A.3 (Borisov and Hua, 2009 [BH09, Proposition 4.1]). We have

$$H^0(\mathfrak{X}_N, \mathcal{O}_{\mathfrak{X}_N}(D)) \cong \bigoplus_{m \in P_D \cap \mathcal{M}} \mathbb{C} \cdot \chi^m.$$

The G-invariant section of $\mathcal{O}_{U_{\Sigma}}$ corresponding to $\chi^m, m \in P_D \cap \mathcal{M}$, is $\prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}}$, where $a_{\rho} = |E_{\rho}|\langle m, u_{\rho} \rangle + b_{\rho}$.

Proof. We have $H^0(U_{\Sigma}, \mathcal{O}_{U_{\Sigma}}) = \mathbb{C}[z_{\rho} : \rho \in \Sigma(1)]$. The global section $\prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}}$ is *G*-invariant if and only if

$$\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}} \cdot \prod_{\rho \in \Sigma(1)} z_{\rho}^{a_{\rho}} = \prod_{\rho \in \Sigma(1)} (z_{\rho} \lambda_{\rho})^{a_{\rho}} \text{ for all } \rho \in \Sigma(1),$$

which is equivalent to the equations $\prod_{\rho \in \Sigma(1)} \lambda_{\rho}^{b_{\rho}-a_{\rho}} = 1$ for all $\rho \in \Sigma(1)$. By exactness of (50), this is equivalent to the existence of $m \in \mathcal{M}$ such that $a_{\rho} - b_{\rho} = |E_{\rho}|\langle m, u_{\rho} \rangle$ for all $\rho \in \Sigma(1)$.

B Combinatorial rules for the linear system of equations \mathbb{V}_{bw}

In this appendix, we collect some combinatorial rules that facilitate the computation of the small polygons and equations in V_{bw} .

B.1 Equivalent description of the small polygons

Consider the lines

$$L_{\rho} := \{ m \in \mathcal{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle = -b_{\rho} \}$$
(53)

that form the boundary of the small Newton polygon N_{bw} . We give an alternate description of these lines. Recall that $\widetilde{\Gamma}$ be the biperiodic graph on the plane given by the lift of Γ to the universal cover of \mathbb{T} . The zig-zag paths in $\widetilde{\Gamma}$ for a given ρ divide the plane into an infinite collection of strips $\mathcal{S}_{\rho}(d)$ parameterized by $d \in \frac{1}{|E_{\rho}|}\mathbb{Z}$ such that

$$\mathcal{S}_{\rho}(d) \cap V(\widetilde{\Gamma}) = \{ \mathbf{v} \in V(\widetilde{\Gamma}) : [D_{\rho}]\mathbf{D}(\mathbf{v}) = d \},\$$

where for a divisor D, $[D_{\rho}]D$ denotes the coefficient of D_{ρ} in D. We assign to each strip $S_{\rho}(d)$ a line $L_{\rho}(d)$ in $M_{\mathbb{R}}$ parallel to E_{ρ} , using the following rule illustrated on Figure 13:

- 1. The line associated to a strip $S_{\rho}(d)$ contains the side E_{ρ} if and only if either
 - i) The strip $S_{\rho}(d)$ is on the right (when facing in the direction of the path) of a zig-zag path $\alpha_1 \in Z_{\rho}$, and α_1 contains b.
 - ii) The strip $S_{\rho}(d)$ contains b, and b is not in a zig-zag path in Z_{ρ} , or
- 2. Moving to the strip to the left shifts the line $1/|E_{\rho}|$ steps to the left.

We call the strip to the left of the one whose line contains E_{ρ} , and all strips obtained by its translations by $H_1(\mathbb{T}, \mathbb{Z})$, exceptional strips.

Proposition B.1. If we associate lines to strips as above, the boundary of the small Newton polygon N_{bw} is given by the lines $\{L_{\rho}(d_{\rho})\}$, where $d_{\rho} \in \frac{1}{|E_{\rho}|}\mathbb{Z}$ is determined by the condition $w \in S_{\rho}(d_{\rho})$, that is, it is the index of the strip containing w in the direction ρ .

Proof. In order for the line L_{ρ} in (53) to contain E_{ρ} , we must have $b_{\rho} = 0$, where b_{ρ} is the coefficient of D_{ρ} in (20). We have to consider two cases.

1. There is a zig-zag path $\alpha \in Z_{\rho}$ such that b is contained in α . We need $[D_{\rho}](\mathbf{D}(b)) = \frac{1}{|E_{\rho}|} + [D_{\rho}](\mathbf{D}(w))$, which means w is contained in the strip S to the right of the one containing b, with α separating the two strips.



Figure 13: The lifts of zig-zag paths $\alpha_1, \ldots, \alpha_k$ in Z_{ρ} divide the plane into strips. The side L_{ρ} of the small polygon N_{bw} and the columns of the matrix \mathbb{V}_{bw} are determined by the strips containing b and w. The black vertex b is the black dot. On the left panel, b is on a zig-zag path, and on the right, it is between two zig-zag paths. Written inside each strip in blue is the subset of Z_{ρ} that gives rise to equations in \mathbb{V}_{bw} if w is contained in that strip. Exceptional strips are shaded.

2. No zig-zag path in Z_{ρ} contains b. In this case, we need the coefficients $[D_{\rho}](\mathbf{D}(\mathbf{b})) = [D_{\rho}](D(\mathbf{w}))$, which means w is in the strip S containing b.

If w_2 is a white vertex in the strip to the left of the strip containing a white w_1 , then $[D_\rho](\mathbf{D}(w_2)) = [D_\rho](\mathbf{D}(w_1)) + \frac{1}{|E_\rho|}$. So if we define $b_\rho(w_1)$ and $b_\rho(w_2)$ as in (20) with $w = w_1$ and $w = w_2$ respectively, then $b_\rho(w_2) = b_\rho(w_1) + \frac{1}{|E_\rho|}$. Note that the line (53) which bounds N_{bw_2} is given by

$$L_{\rho}(\mathbf{w}_2) := \{ m \in \mathbf{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle = b_{\rho}(\mathbf{w}_2) \}.$$

The similar line which bounds N_{bw_1} is

$$L_{\rho}(\mathbf{w}_1) := \{ m \in \mathbf{M}_{\mathbb{R}} : \langle m, u_{\rho} \rangle = b_{\rho}(\mathbf{w}_1) \},\$$

so the line $L_{\rho}(\mathbf{w}_2)$ is obtained from the line $L_{\rho}(\mathbf{w}_1)$ by shifting $1/|E_{\rho}|$ steps to the left.

B.2 The equations in \mathbb{V}_{bw}

We describe the equations of type 2 in Section 3.1.2.

Let $\rho \in \Sigma(1)$ be a ray and let $Z_{\rho} = \{\alpha_1, \ldots, \alpha_k\}$, where $\alpha_1, \ldots, \alpha_k$ are labeled in cyclic order. Their lifts to the universal cover of the torus divides it into strips, see Figure 13. We denote by S_i the strip immediately to the right of α_i .

Proposition B.2. The set of extra linear equations is described as follows:

1. One of these zig-zag paths contains b. We can assume it is α_1 . Then the subset of Z_{ρ} that contributes an equation to \mathbb{V}_{bw} is:

$$empty \ if \ w \in S_k;$$

$$\alpha_{i+1}, \dots, \alpha_k \ if \ w \in S_i \ for \ some \ i \neq k.$$
(54)

2. The vertex b is not in any of zig-zag paths in Z_{ρ} . Then the subset of Z_{ρ} is

$$\alpha_1, \dots, \alpha_k \text{ if } w \in S_k;$$

$$\alpha_{i+1}, \dots, \alpha_k \text{ if } w \in S_i, \text{ for some } i \neq k.$$
(55)

Proof. Plugging

$$Y_{\mathrm{bw}} = D_N - \mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho}.$$

into (44), we first observe that (44) does not change if we replace w by its any translate on the universal cover because $\mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b})$ changes by the same amount as $[E_{\rm bw}]|_{\mathcal{C}}$ but with the opposite sign. Therefore we may assume that among all its possible translates in the universal cover, the strip S_i is the one that is immediately to the right of b. Then we have

$$(\mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}))\Big|_{\mathcal{C} \cap D_a} = -\nu(\alpha_1) - \dots - \nu(\alpha_i)$$

and the coefficient of D_{ρ} in $(\mathbf{D}(w) - \mathbf{D}(b))$ is $-\frac{i}{k}$.

Now we distinguish two cases:

1. Suppose b is contained in α_1 , so that the coefficient of D_{ρ} in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho}$ is $\frac{1}{k}$. Then we have

$$-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \right\|_{\mathcal{C} \cap D_{\rho}} = 0,$$

so that (44) is $\nu(\alpha_{i+1}) + \cdots + \nu(\alpha_k)$, which proves (54).

2. Suppose b is not contained in any of the α_i , so that the coefficient of D_{ρ} in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: b \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho}$ is 0. We have

$$-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_{\rho}: \mathbf{b} \in \alpha} \frac{1}{|E_{\rho}|} D_{\rho} \right\| \Big|_{\mathcal{C} \cap D_{\rho}} = \begin{cases} 0 & \text{if } i \neq k, \\ \sum_{j=1}^{k} \nu(\alpha_{j}) & \text{if } i = k. \end{cases}$$

This gives

(44) =
$$\begin{cases} \nu(\alpha_{i+1}) + \dots + \nu(\alpha_k) & \text{if } i \neq k, \\ \sum_{j=1}^k \nu(\alpha_j) & \text{if } i = k. \end{cases}$$

We obtain (55).

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