

C-NETWORKS AND THE PLANAR ISING INVERSE PROBLEM

YASSINE ELMAAZOUZ AND TERRENCE GEORGE

ABSTRACT. We solve the inverse problem for Ising models on reduced planar graphs in a disk, i.e., recovering the edge coupling constants from the boundary spin correlations. A recursive solution to this problem was provided by Galashin–Pylyavskyy. Our solution is non-recursive. It is based on an Ising analog of the chamber ansatz which asserts that the inverse map should factor through variables on the graph that transform under the Ising Y- Δ move according to the discrete CKP equation.

1. INTRODUCTION

An *Ising model* in a disk $\mathbb{D}$ consists of a planar graph $G = (V, E, F)$ embedded in $\mathbb{D}$ with n boundary vertices labeled $b_1^\partial, \dots, b_n^\partial$ in clockwise cyclic order together with a function $J : E \rightarrow \mathbb{R}_{>0}$ called the *coupling constant*. A *spin configuration* is a function $\sigma : V \rightarrow \{-1, 1\}$ assigning to each vertex $v \in V$ a spin σ_v . The probability of a spin configuration is defined as

$$\mathbf{P}(\sigma) := \frac{1}{Z} \prod_{e=uv \in E} e^{J_e \sigma_u \sigma_v}, \quad \text{where} \quad Z := \sum_{\sigma: V \rightarrow \{-1, 1\}} \prod_{e=uv \in E} e^{J_e \sigma_u \sigma_v}$$

is the *partition function*. We denote by $[n]$ the finite set $[n] := \{1, \dots, n\}$ and by $\binom{[n]}{k}$ the set of all subsets of $[n]$ of size k . Given $i, j \in [n]$, we define the *boundary correlation*

$$\langle \sigma_i \sigma_j \rangle := \sum_{\sigma: V \rightarrow \{-1, 1\}} \mathbf{P}(\sigma) \sigma_{b_i^\partial} \sigma_{b_j^\partial}.$$

These correlations are organized into an $n \times n$ symmetric real matrix with 1s on the diagonal

$$M(G, J) := (\langle \sigma_i \sigma_j \rangle)_{i, j \in [n]},$$

called the *boundary correlation matrix*. We write

$$\text{Corr}_G : \mathcal{I}_G^{>0} \rightarrow \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1), \quad J \mapsto M(G, J),$$

where

$$\mathcal{I}_G^{>0} := \left\{ J : E(G) \rightarrow \mathbb{R}_{>0} \right\}$$

is the space of Ising models on the graph G and $\text{Mat}_n^{\text{sym}}(\mathbb{R}, 1)$ denotes the space of real symmetric $n \times n$ matrices with 1s on the diagonal. The *inverse problem for the Ising model* asks to recover the coupling constant J from the boundary correlation matrix $M(G, J)$.

In order for this inverse problem to have a unique solution, one must impose a reducedness assumption on G . This is analogous to the situation for electrical networks. Lam [24] showed that reduced electrical networks in a disk, modulo certain local transformations of the graph called Y- Δ moves, are classified by matchings of $[2n]$. In the Ising setting, the same combinatorial object appears: to a reduced graph G one associates a matching π of $[2n]$.

Throughout the paper, G will be assumed reduced, and all constructions will be compatible with Y- Δ moves.

Galashin–Pylyavskyy [12] gave a recursive solution to the Ising inverse problem using operations called adjoining a boundary edge and adjoining a boundary spike. These operations are the Ising analogs of operations for electrical networks introduced by Curtis–Ingerman–Morrow [9]. The purpose of this paper is to give a direct and non-recursive solution to the Ising inverse problem using ideas that originate in the chamber ansatz of Berenstein–Fomin–Zelevinsky [3], later developed for the positive Grassmannian by Marsh–Scott [27], extended to arbitrary positroid cells by Muller–Speyer [29], and applied to electrical networks in the big cell by George [14]. In this paper, we develop an Ising analog of this chamber-ansatz framework.

We define two auxiliary functions $s, c : E(G) \rightarrow (0, 1)$ as follows:

$$s_e := \operatorname{sech}(2J_e), \quad c_e := \tanh(2J_e).$$

Rather than solving directly for the coupling constant J , we solve for the functions s and c from which we can recover J as $\frac{1}{2} \operatorname{arctanh}(c)$. The chamber ansatz for the Ising model asks that these functions factor through a collection of positive variables attached to the vertices and faces of G . More precisely, a C -network on G is a function

$$C : V(G) \sqcup F(G) \rightarrow \mathbb{R}_{>0}$$

defined modulo rescaling by $\mathbb{R}_{>0}$. Let

$$\mathcal{C}_G^{>0} \cong \mathbb{R}_{>0}^{|V(G)|+|F(G)|-1}$$

be the space of C -networks with underlying graph G . In terms of the C -network, the functions s and c are given by

$$(1) \quad s_e := \sqrt{\frac{C_f C_g}{C_u C_v + C_f C_g}}, \quad c_e := \sqrt{\frac{C_u C_v}{C_u C_v + C_f C_g}},$$

where $e = uv$ and f, g are the two faces adjacent to e . In other words, (1) defines a map

$$q_G : \mathcal{C}_G^{>0} \rightarrow \mathcal{I}_G^{>0}.$$

There is a positive torus

$$(2) \quad T_\pi^{>0} \subset \mathbb{R}_{>0}^{2n},$$

depending only on the matching π , that acts on $\mathcal{C}_G^{>0}$. The map q_G is invariant under this action, and therefore descends to a map

$$q_G : \mathcal{C}_G^{>0} / T_\pi^{>0} \longrightarrow \mathcal{I}_G^{>0}.$$

In this framework, solving the inverse problem reduces to constructing the dotted map making the following diagram commute:

$$(3) \quad \begin{array}{ccc} \mathcal{C}_G^{>0} / T_\pi^{>0} & \xrightarrow{q_G} & \mathcal{I}_G^{>0} \\ & \swarrow \text{?} & \nearrow \operatorname{Corr}_G^{-1} \\ & \{\text{Boundary correlation matrices}\} \subset \operatorname{Mat}_n^{\operatorname{sym}}(\mathbb{R}, 1) & \end{array}$$

The terminology C -network is motivated by the transformation rule for the variables C under the Y - Δ move. This transformation is the discrete CKP equation in the form introduced by Kashaev in his study of the Ising Y - Δ move [20]; see also Schief [31]. It is Cayley's $2 \times 2 \times 2$ hyperdeterminant [13] and appears in the work of Holtz–Sturmfels [18] on the principal minor assignment problem for symmetric matrices. Related incarnations of the same equation appear in the work of Kenyon–Pemantle [22, 21], Leaf [25], Melotti [28], and Arthamonov–Harnad–Hurtubise [2], where it is connected to statistical mechanics, cluster algebras, and Grassmannians. It also appears in the work of Doliwa on discrete integrable systems [10] and in discrete differential geometry in the work of Bobenko–Schief [4, 5]. The chamber ansatz formula (1) appears in Melotti [28, Remark 3] and, up to a sign, in the formula (10) for the partial correlations in Sturmfels–Tsukerman–Williams [34].

We will construct the dotted map in (3) via a modification of the twist map of Muller–Speyer [29] for positroid cells. We first recall how boundary correlation matrices are related to the totally nonnegative Grassmannian. The *Grassmannian* $\text{Gr}_{n,2n}$ is the space of n -dimensional subspaces of $\mathbb{R}^{2n}$. Such a subspace $V \subset \mathbb{R}^{2n}$ can be represented as the rowspan of a full-rank $n \times 2n$ matrix. Its maximal minors

$$\Delta_I(V), \quad I \in \binom{[2n]}{n},$$

are homogeneous coordinates on $\text{Gr}_{n,2n}$, called *Plücker coordinates*. The *totally nonnegative Grassmannian* is the subset

$$\text{Gr}_{n,2n}^{\geq 0} := \left\{ V \in \text{Gr}_{n,2n} : \Delta_I(V) \geq 0 \text{ for all } I \in \binom{[2n]}{n} \right\}.$$

Postnikov [30] showed that $\text{Gr}_{n,2n}^{\geq 0}$ decomposes into *positroid cells*

$$\text{Gr}_{n,2n}^{\geq 0} = \bigsqcup_{\mathcal{M}} \Pi_{\mathcal{M}}^{\geq 0}, \quad \text{where} \quad \Pi_{\mathcal{M}}^{\geq 0} := \left\{ V \in \text{Gr}_{n,2n}^{\geq 0} : \Delta_I(V) > 0 \text{ if and only if } I \in \mathcal{M} \right\}.$$

The indexing sets $\mathcal{M} \subset \binom{[2n]}{n}$ that arise in this way are called *positroids*. Equivalently, positroid cells are indexed by (decorated) permutations of $[2n]$; we denote the positroid corresponding to a permutation π by $\mathcal{M}_{\pi}$.

Galashin–Pylyavskyy [12, Section 2.3] define a map

$$\phi_n : \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1) \rightarrow \text{Gr}_{n,2n},$$

called the *doubling map*, whose image lies in the orthogonal Grassmannian

$$\text{OGr}_{n,2n} := \left\{ V \in \text{Gr}_{n,2n} : \Delta_I(V) = \Delta_{[2n] \setminus I}(V) \text{ for all } I \in \binom{[2n]}{n} \right\}.$$

Moreover, if G is reduced and π is the associated matching of $[2n]$, then π determines a positroid $\mathcal{M}_{\pi}$ (viewing the matching π as a permutation of $[2n]$), and

$$\phi_n(M(G, J)) \in \text{OGr}_{n,2n} \cap \Pi_{\mathcal{M}_{\pi}}^{\geq 0}.$$

Next, we relate C -networks to the totally nonnegative Grassmannian. C -networks on graphs G with the maximal matching (i.e., $i \mapsto i + n \pmod{2n}$) were first defined by Kenyon–Pemantle [21], who used them to parameterize the space of positive-definite symmetric matrices. Here, the C -variables are principal and almost-principal minors of the matrix. Equivalently, this space can be identified with the positive part $\text{LG}_{n,2n}^{\geq 0}$ of a Lagrangian

Grassmannian $\text{LG}_{n,2n} \subset \text{Gr}_{n,2n}$ with respect to the symplectic form

$$\omega(x, y) := \sum_{i=1}^n (-1)^{i-1} (x_i y_{i+n} - x_{i+n} y_i), \quad \text{for } x, y \in \mathbb{R}^{2n}.$$

This semialgebraic set appears in work of Shevchenko [33], and differs from the totally positive Lagrangian Grassmannian (in Lusztig's sense [26]) studied by Karpman [19] and the totally positive Lagrangian Grassmannian arising in the theory of electrical networks in work of Lam [24], Chepuri–George–Speyer [8] and Bychkov–Gorbounov–Kazakov–Talalaev [7], which use other symplectic forms. Under this identification with $\text{LG}_{n,2n}^{>0}$, the C -variables, i.e., the principal and almost-principal minors, are Plücker coordinates of the point in $\text{LG}_{n,2n}^{>0}$ and this construction can be viewed as a specialization of a construction of Scott [32] for the totally nonnegative Grassmannian. Generalizing this, we define subsets

$$\Lambda_\pi^{>0} \subset \Pi_{\mathcal{M}_\pi}^{>0},$$

which we call *locally Lagrangian positroid cells*, together with maps

$$\mathbf{H}_G : \Lambda_\pi^{>0} \rightarrow \mathcal{C}_G^{>0},$$

where each C -variable is given by a Plücker coordinate of the point in $\Lambda_\pi^{>0}$. We show that the conditions defining $\Lambda_\pi^{>0}$ inside $\Pi_{\mathcal{M}_\pi}^{>0}$ may be viewed as local Lagrangian conditions, thereby justifying the terminology. We then prove that $\mathbf{H}_G$ is a bijection and that these bijections are compatible with Y - Δ moves.

Remark 1.1. A different collection of subsets of the Grassmannian generalizing the big cell of the Lagrangian Grassmannian appears in the study of Gaussoids by Boege–D’Alì–Kahle–Sturmfels [6].

The final ingredient is a modification of the twist map. For an arbitrary positroid $\mathcal{M}$, Muller–Speyer [29] define right and left twist maps

$$\vec{\tau}_{\mathcal{M}}, \overleftarrow{\tau}_{\mathcal{M}} : \Pi_{\mathcal{M}}^{>0} \rightarrow \Pi_{\mathcal{M}}^{>0},$$

which are inverse to each other and which relate the A - and X -cluster coordinates of the positroid cell. In the Ising setting, the twist maps must be modified by the natural action of $\mathbb{R}_{>0}^{2n}$ rescaling the columns of a matrix representative to be compatible with q_G ; a similar modification was required for electrical networks in George [14].

The positive torus (2) acts on $\Lambda_\pi^{>0}$, and the map $\mathbf{H}_G$ is equivariant and therefore descends to a map

$$\mathbf{H}_G : \Lambda_\pi^{>0} / T_\pi^{>0} \rightarrow \mathcal{C}_G^{>0} / T_\pi^{>0}.$$

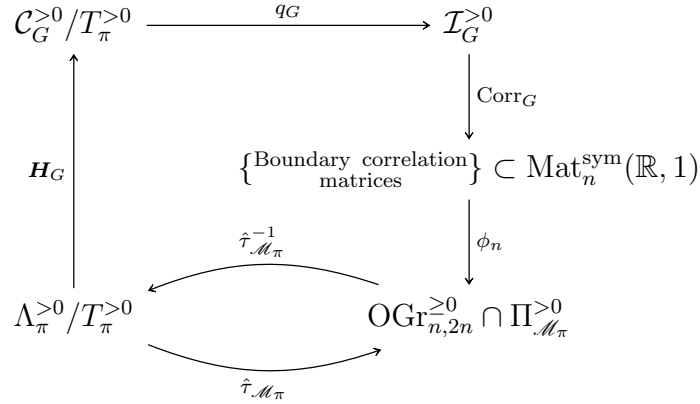
The *Ising twist and inverse twist maps*

$$\hat{\tau}_{\mathcal{M}_\pi} : \Lambda_\pi^{>0} / T_\pi^{>0} \rightarrow \text{OGr}_{n,2n} \cap \Pi_{\mathcal{M}_\pi}^{>0}, \quad \hat{\tau}_{\mathcal{M}_\pi}^{-1} : \text{OGr}_{n,2n} \cap \Pi_{\mathcal{M}_\pi}^{>0} \rightarrow \Lambda_\pi^{>0} / T_\pi^{>0}.$$

defined after passing to the quotient $\Lambda_\pi^{>0} / T_\pi^{>0}$. We can now state our main theorem.

Theorem 1.2. *The Ising twist map $\hat{\tau}_{\mathcal{M}_\pi} : \Lambda_\pi^{>0} / T_\pi^{>0} \rightarrow \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}$ is a bijection whose inverse is $\hat{\tau}_{\mathcal{M}_\pi}^{-1}$. They fit into the following commutative diagram in which every map is a*

bijection:



In particular, the dotted map in (3) which solves the inverse problem is given by the composition

$$H_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1} \circ \phi_n.$$

Open problems. We end the introduction with some open problems that we believe are interesting for further study.

Problem 1.3. It is natural to study the algebro-geometric counterparts of locally Lagrangian positroid cells, i.e., the Zariski closures of the semialgebraic sets $\Lambda_\pi^{>0}$ inside the complex Lagrangian Grassmannian, similarly to the positroid varieties of Knutson–Lam–Speyer [23].

Problem 1.4. The chamber ansatz formula (1) also appears in formula (10) for the partial correlations in Sturm–Tsukerman–Williams [34]. This suggests that the two settings may be related and it would be very interesting to understand the precise nature of this relation.

Problem 1.5. It would be interesting to find the electrical-network analogs of the locally Lagrangian positroid cells. For the maximal matching, this is a totally positive orthogonal Grassmannian; see Henriques–Speyer [17] and George [14].

Problem 1.6. The target space of the inverse map $\hat{\tau}_{\mathcal{M}_\pi}^{-1}$ is the quotient $\Lambda_\pi^{>0}$ of the locally Lagrangian positroid cell modulo the action of $T_\pi^{>0}$. When π is the matching $\pi_0: i \mapsto i + n \pmod{2n}$, the cell $\Lambda_{\pi_0}^{>0}$ is the big cell $\text{LG}_{n,2n}^{>0}$ of the Lagrangian Grassmannian in [33]. It would be interesting to study the quotient $\Lambda_{\pi_0}^{>0}/T_{\pi_0}^{>0} = \text{LG}_{n,2n}^{>0}/T_{\pi_0}^{>0}$ in light of [1].

Organization of the paper. In Section 2, we review the necessary background on the positive Grassmannian and its connections to the dimer model and A -networks. In Section 3, we recall the corresponding background on the positive orthogonal Grassmannian and its relation to the Ising model. In Section 4, we introduce C -networks and locally Lagrangian positroid cells. Finally, in Section 5, we define the maps appearing in the diagram of the main result, Theorem 1.2, and prove the theorem.

2. BACKGROUND ON THE POSITIVE GRASSMANNIAN, THE DIMER MODEL AND A -NETWORKS

In this section we collect some background on the totally nonnegative Grassmannian which we shall need throughout this article.

2.1. The Grassmannian and positivity. Let $1 \leq k \leq n$ be positive integers. The real *Grassmannian* $\text{Gr}_{k,n} := \text{Gr}_{k,n}(\mathbb{R})$ is the moduli space of k -dimensional subspaces of the vector space $\mathbb{R}^n$. A k -dimensional subspace $V \subset \mathbb{R}^n$ can be represented as the rowspan of a full-rank $k \times n$ matrix M , and the map

$$\text{Gr}_{k,n} \rightarrow \mathbb{P}^{\binom{n}{k}-1}, \quad V = \text{rowspan}(M) \mapsto (\Delta_I(V) = \det(M_I))_{I \in \binom{[n]}{k}},$$

is the *Plücker embedding* of the Grassmannian $\text{Gr}_{k,n}$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$. Here M_I denotes the $k \times k$ -submatrix of M with all rows but columns only from the index set I . The Grassmannian $\text{Gr}_{k,n}$ is a projective variety and the $\Delta_I(V)$ are called *Plücker coordinates*. We will often omit the vector space in the notation and simply write Δ_I . The *totally positive Grassmannian* $\text{Gr}_{k,n}^{>0}$ is the semialgebraic subset of points in $\text{Gr}_{k,n}$ such that $\Delta_I > 0$ for all $I \in \binom{[n]}{k}$. Its topological closure in $\mathbb{P}^{\binom{n}{k}-1}$ is the *totally nonnegative Grassmannian* consisting of points in $\text{Gr}_{k,n}^{\geq 0}$ such that $\Delta_I \geq 0$ for any $I \in \binom{[n]}{k}$. Given a point $V \in \text{Gr}_{k,n}^{\geq 0}$, its *matroid* $\mathcal{M}_V$ is defined by

$$\mathcal{M}_V := \left\{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \right\}.$$

In this article, we shall identify any rank k matroid $\mathcal{M}$ on the ground set $[n]$ with its set of bases and write $\mathcal{M} \subset \binom{[n]}{k}$. Any matroid $\mathcal{M} \subset \binom{[n]}{k}$ such that there exists a $V \in \text{Gr}_{k,n}^{\geq 0}$ with $\mathcal{M} = \mathcal{M}_V$ is called a *positroid*. The corresponding *positroid cell* is

$$\Pi_{\mathcal{M}}^{\geq 0} := \{ V \in \text{Gr}_{k,n}^{\geq 0} : \mathcal{M}_V = \mathcal{M} \}.$$

By definition of positroid cells, we have

$$(4) \quad \text{Gr}_{k,n}^{\geq 0} = \bigsqcup_{\text{positroids } \mathcal{M}} \Pi_{\mathcal{M}}^{\geq 0},$$

where the union is taken over all positroids of rank k on the ground set $[n]$.

Torus actions will play an important role in this paper and we will systematically keep track of actions on the various objects we define. In particular, we note that $\mathbb{R}_{>0}^n$ acts on $\text{Gr}_{k,n}^{\geq 0}$ by rescaling the basis vectors of $\mathbb{R}^n$, i.e., by rescaling the columns of any matrix representative. More explicitly, we have

$$(t \cdot \Delta)_I = \left(\prod_{i \in I} t_i \right) \Delta_I, \quad \text{for } \Delta \in \text{Gr}_{k,n}^{\geq 0} \text{ and } t = (t_1, \dots, t_n) \in \mathbb{R}_{>0}^n,$$

and this action preserves each positroid cell $\Pi_{\mathcal{M}}^{\geq 0}$.

2.2. Plabic graphs in the disk. A *plabic graph* Γ is a bipartite graph embedded in the disk $\mathbb{D}$ such that:

- (1) Γ has exactly n degree-1 boundary vertices, labeled $u_1^\partial, \dots, u_n^\partial$ in clockwise cyclic order, each of which is adjacent to exactly one internal vertex.
- (2) The internal vertices are colored black or white and adjacent vertices have opposite colors.

Although there is a unique choice of coloring of boundary vertices that makes Γ bipartite, it will be more convenient to treat the boundary vertices as uncolored. We denote the vertices, edges and faces of Γ by $V(\Gamma)$, $E(\Gamma)$ and $F(\Gamma)$ respectively. We write $B(\Gamma)$, $W(\Gamma)$ and $\partial V(\Gamma)$ for the black, white and boundary vertices of Γ respectively, so $V(\Gamma) = B(\Gamma) \sqcup W(\Gamma) \sqcup \partial V(\Gamma)$.

We denote by $f_{i,i+1}^\partial$ the boundary face between u_i^∂ and u_{i+1}^∂ . We denote by

$$k := |W(\Gamma)| - |B(\Gamma)| + \left| \{b \in B(\Gamma) : b \text{ is adjacent to the boundary}\} \right|$$

the *helicity* of Γ and say that Γ is of *type* (k, n) .

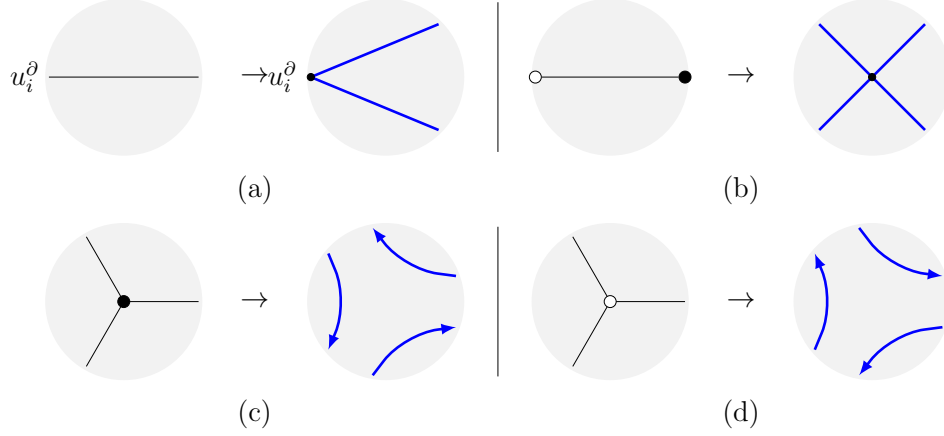


FIGURE 1. The construction of the oriented medial graph $\Gamma^\times$ from Γ at (a) boundary edges, (b) internal edges, (c) black vertices and (d) white vertices. The vertices may be of any degree.

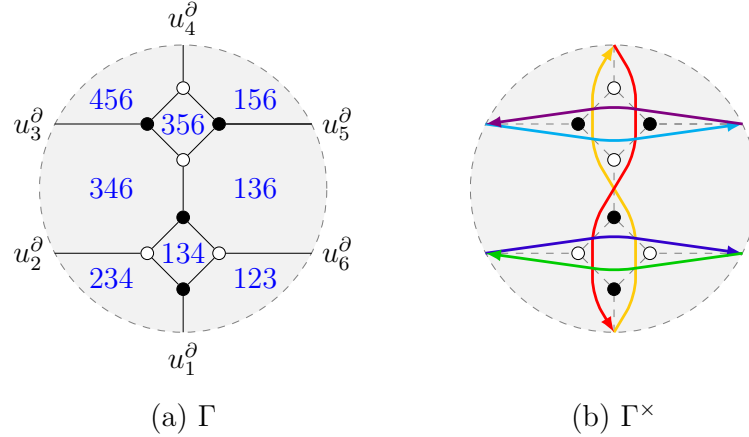


FIGURE 2. Example of the medial graph $\Gamma^\times$ and target face labels of a plabic graph Γ .

Let $\Gamma^\times$ denote the *directed medial graph* of Γ constructed as illustrated in Figure 1. The boundary vertices of $\Gamma^\times$ naturally correspond to the boundary vertices of Γ ; we therefore label them by $u_1^\partial, \dots, u_n^\partial$ as well. A *strand* of Γ is a maximal directed path in $\Gamma^\times$. Each strand is either a closed loop or a directed path connecting two boundary vertices. The *decorated permutation* $\pi_\Gamma = (\pi_\Gamma, \text{col})$ consists of:

- (1) A permutation $\pi_\Gamma : [n] \rightarrow [n]$ defined by

$$\pi_\Gamma(i) := j, \quad \text{if the strand starting at } u_i^\partial \text{ ends at } u_j^\partial.$$

- (2) When i is a fixed point of π_Γ , a *color* $\text{col}(i) \in \{\bullet, \circ\}$. The color will not play a role for the plabic graphs that arise from Ising models that we treat in this paper; hence, we omit the details of its construction.

We say that the graph Γ is *reduced* if the following hold:

- (1) No strand is a closed loop.
- (2) No strand has a self-intersection.
- (3) No two strands form a *parallel bigon*, i.e., there are no two edges $e_1, e_2 \in E(\Gamma)$ such that both strands go through e_1 and e_2 with the same orientation.

Hereafter, we assume that our plabic graphs are reduced.

Given a reduced plabic graph Γ of type (k, n) , we define two maps

$$\mathbf{S}, \mathbf{T} : F(\Gamma) \rightarrow \binom{[n]}{k},$$

called *source* and *target labels* respectively, as follows. Each strand of Γ is assigned a label $i \in [n]$, determined by its source vertex or its target vertex, depending on the chosen convention. Since Γ is reduced, there are exactly n such strands labeled by $[n]$. Each face f of Γ is labeled by the subset of strand labels such that f lies on the left side of the strand. It is easy to see that each face of Γ thus obtains a label $I \subset [n]$ of the same cardinality k .

A *Grassmann necklace* $\mathbf{I} = (I_1, \dots, I_n)$ is a sequence of k -element subsets of $[n]$ such that for each $i \in [n]$, there is a $j_i \in [n]$ such that $I_{i+1} = I_i \setminus \{i\} \cup \{j_i\}$. The *Grassmann necklace* $\mathbf{I}_\Gamma = (I_1, \dots, I_n)$ of Γ is defined by $I_i := \mathbf{T}(f_{i-1,i}^\partial)$. Here $f_{i-1,i}^\partial \in F(\Gamma)$ denotes the boundary face of Γ between the boundary vertices u_{i-1}^∂ and u_i^∂ . Similarly, there is a dual object called the *reverse Grassmann necklace* defined by $\mathbf{I}_\Gamma^{\text{rev}} = (I_1^{\text{rev}}, \dots, I_n^{\text{rev}})$ where $I_i^{\text{rev}} := \mathbf{S}(f_{i,i+1}^\partial)$.

Example 2.1. Figure 2 shows a plabic graph Γ with $\pi_\Gamma = 465132$ (in one-line notation) together with its medial graph $\Gamma^\times$ and target face labels, from which we get that the Grassmann necklace

$$\mathbf{I}_\Gamma = (123, 234, 346, 456, 156, 136).$$

A *dimer cover* or *almost perfect matching* of Γ is a subset of the edges that uses every internal vertex exactly once and every boundary vertex at most once. The *boundary* of a dimer cover M is defined as follows:

$$\partial M := \left\{ i \in [n] : \begin{array}{l} u_i^\partial \text{ is adjacent to a white vertex and covered by } M, \\ \text{or is adjacent to a black vertex and not covered by } M \end{array} \right\}.$$

The *positroid* $\mathcal{M}_\Gamma$ of Γ is then defined as

$$\mathcal{M}_\Gamma := \left\{ I \in \binom{[n]}{k} : \text{there is a dimer cover } M \text{ with } \partial M = I \right\}.$$

The local modifications of plabic graphs shown in Figure 4 are called *moves*. Two plabic graphs Γ and Γ' are said to be *move-equivalent* if they are related by a sequence of moves.

Theorem 2.2 (Postnikov [30]). *The maps $\Gamma \mapsto \pi_\Gamma$, $\Gamma \mapsto \mathbf{I}_\Gamma$ and $\Gamma \mapsto \mathcal{M}_\Gamma$ that associate to a reduced plabic graph Γ (up to move-equivalence) its decorated permutation, Grassmann necklace and its positroid, respectively, are all bijections. In particular, decorated permutations, Grassmann necklaces and positroids all classify move-equivalence classes of reduced plabic graphs.*

2.3. Dimer models. An *edge weight* on a plabic graph Γ is a function

$$\text{wt} : E(\Gamma) \rightarrow \mathbb{R}_{>0}.$$

We say that two edge weights wt_1 and wt_2 are *gauge equivalent* if there exists a function

$$g : V(\Gamma) \rightarrow \mathbb{R}_{>0}$$

satisfying $g(u_i^\partial) = 1$ for $i \in [n]$ such that

$$\text{wt}_1(e) = g(b)^{-1} \text{wt}_2(e) g(w) \quad \text{for all } e = bw \in E(\Gamma).$$

We denote the gauge equivalence class of a weight wt by $[\text{wt}]$. For any face f of Γ delimited by a sequence of edges

$$b_1 \xrightarrow{e_1} w_1 \xrightarrow{e_2} b_2 \xrightarrow{e_3} \dots \xrightarrow{e_{2k-2}} b_k \xrightarrow{e_{2k-1}} w_k \xrightarrow{e_{2k}} b_1,$$

the function

$$X_f(\text{wt}) := \frac{\text{wt}(e_1) \cdots \text{wt}(e_{2k-1})}{\text{wt}(e_2) \cdots \text{wt}(e_{2k})}$$

is gauge-invariant. These functions are called *X-cluster variables*.

A *dimer model* is a pair $(\Gamma, [\text{wt}])$ where Γ is a reduced plabic graph and $[\text{wt}]$ is a gauge equivalence class of edge weights on Γ . The *X-cluster variables* provide coordinates on the space of edge weights modulo gauge equivalence up to the unique relation $\prod_{f \in F(\Gamma)} X_f = 1$. Let

$$\mathcal{X}_\Gamma^{>0} \cong \mathbb{R}_{>0}^{|F(\Gamma)|-1}$$

denote the *space of dimer models* with underlying graph Γ . Each move $\Gamma \rightarrow \Gamma'$ induces a bijection between the corresponding spaces of dimer models (see Figure 3)

$$\mathcal{X}_\Gamma^{>0} \xrightarrow{\sim} \mathcal{X}_{\Gamma'}^{>0}.$$

Given a positroid $\mathcal{M}$, the *space of dimer models with positroid $\mathcal{M}$* is defined as

$$\mathcal{X}_\mathcal{M}^{>0} := \bigsqcup_{\Gamma: \mathcal{M}_\Gamma = \mathcal{M}} \mathcal{X}_\Gamma^{>0} / \text{moves}.$$

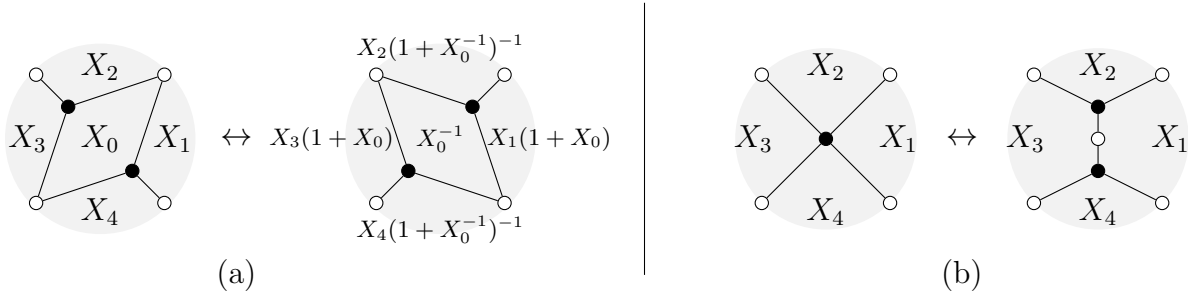


FIGURE 3. Transformation of *X*-cluster variables under (a) spider move and (b) contraction-uncontraction move.

Let wt be an edge weight. For $I \in \binom{[n]}{k}$, let

$$Z_I(\text{wt}) := \sum_{M: \partial M = I} \prod_{e \in M} \text{wt}(e)$$

denote the *partition function* for dimer covers with boundary I . The map

$$\text{Meas}_\Gamma : \mathcal{X}_\Gamma^{>0} \rightarrow \mathbb{P}^{\binom{n}{k}-1}, \quad [\text{wt}] \mapsto (Z_I(\text{wt}))_{I \in \binom{[n]}{k}}$$

is called *boundary measurement*. Here, wt denotes any edge weight representing the gauge equivalence class $[\text{wt}]$. Different choices of representatives result in the same vector $(Z_I(\text{wt}))_{I \in \binom{[n]}{k}}$ modulo rescaling, and therefore the point $(Z_I(\text{wt}))_{I \in \binom{[n]}{k}} \in \mathbb{P}^{\binom{n}{k}-1}$ is the same.

Theorem 2.3 (Postnikov [30]). *Let Γ be a reduced plabic graph and let $\mathcal{M} = \mathcal{M}_\Gamma$ be its positroid. Then*

$$\text{Meas}_\Gamma : \mathcal{X}_\Gamma^{>0} \xrightarrow{\sim} \Pi_{\mathcal{M}}^{>0}$$

is a bijection. Moreover, Meas_Γ is compatible with moves: for any move $\Gamma \rightarrow \Gamma'$, the diagram

$$\begin{array}{ccc} \mathcal{X}_\Gamma^{>0} & \xrightarrow{\sim} & \mathcal{X}_{\Gamma'}^{>0} \\ \text{Meas}_\Gamma \searrow & & \swarrow \text{Meas}_{\Gamma'} \\ & \Pi_{\mathcal{M}}^{>0} & \end{array}$$

commutes. Therefore, the maps $(\text{Meas}_\Gamma)_{\Gamma: \mathcal{M}_\Gamma = \mathcal{M}}$ glue together to a bijection

$$\text{Meas}_{\mathcal{M}} : \mathcal{X}_{\mathcal{M}}^{>0} \xrightarrow{\sim} \Pi_{\mathcal{M}}^{>0}.$$

Consider the following action of $\mathbb{R}_{>0}^n$ on $\mathcal{X}_\Gamma^{>0}$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_{>0}^n$,

$$\lambda \cdot \text{wt}(e) := \begin{cases} \lambda_i \text{wt}(e), & \text{if } e \text{ is the boundary edge incident to } u_i^\partial \text{ and a white internal vertex,} \\ \lambda_i^{-1} \text{wt}(e), & \text{if } e \text{ is the boundary edge incident to } u_i^\partial \text{ and a black internal vertex,} \\ \text{wt}(e), & \text{otherwise.} \end{cases}$$

The action is compatible with moves, and therefore gives an action of $\mathbb{R}_{>0}^n$ on $\mathcal{X}_{\mathcal{M}}^{>0}$. Moreover, the definition is such that Meas_Γ and $\text{Meas}_{\mathcal{M}}$ are equivariant.

2.4. A-networks. An *A-network* is a pair (Γ, A) where Γ is a reduced plabic graph and $A : F(\Gamma) \rightarrow \mathbb{R}_{>0}$ is a function defined modulo rescaling by $\mathbb{R}_{>0}$. Let

$$\mathcal{A}_\Gamma^{>0} \cong \mathbb{R}_{>0}^{|F(\Gamma)|-1}$$

denote the *space of A-networks with underlying plabic graph Γ* .

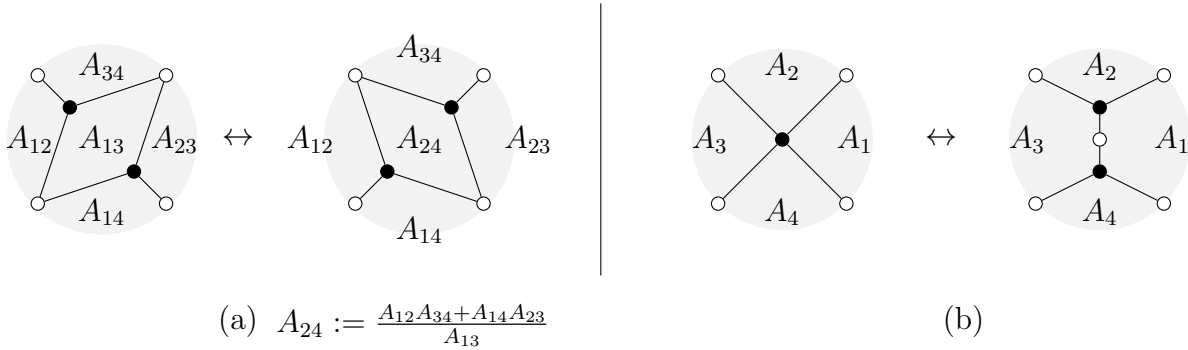


FIGURE 4. Transformation of A -cluster variables under (a) spider move and (b) contraction-uncontraction move.

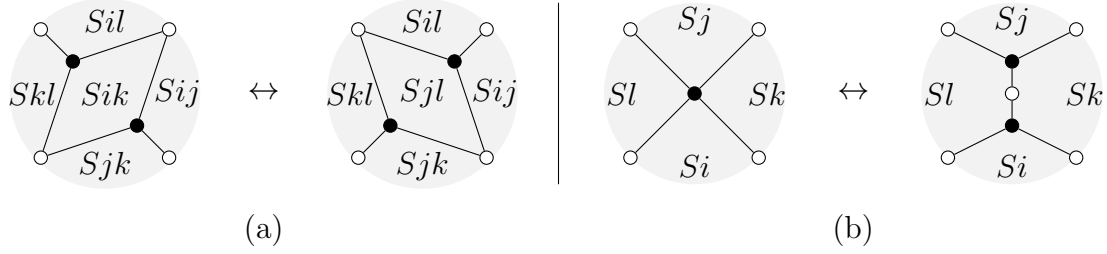


FIGURE 5. (a) Spider move and (b) contraction-uncontraction move with face labels.

Each move $\Gamma \rightarrow \Gamma'$ induces a bijection between the corresponding spaces of A -networks

$$\mathcal{A}_{\Gamma}^{>0} \xrightarrow{\sim} \mathcal{A}_{\Gamma'}^{>0}$$

as shown in Figure 4. Given a positroid $\mathcal{M}$, we define the *space of A -networks with positroid $\mathcal{M}$* to be

$$\mathcal{A}_{\mathcal{M}}^{>0} := \bigsqcup_{\Gamma: \mathcal{M}_{\Gamma} = \mathcal{M}} \mathcal{A}_{\Gamma}^{>0} / \text{moves}.$$

Recall that Plücker coordinates are defined modulo rescaling. For any reduced plabic graph Γ with $\mathcal{M}_{\Gamma} = \mathcal{M}$, we define a map

$$\mathbf{F}_{\Gamma} : \Pi_{\mathcal{M}}^{>0} \rightarrow \mathcal{A}_{\Gamma}^{>0}, \quad V \mapsto (\Delta_{\mathbf{T}(f)}(V))_{f \in F(\Gamma)}$$

assigning to each point in the positroid cell $\Pi_{\mathcal{M}}^{>0}$ an A -network.

The following is due to Scott [32] for the uniform positroid $\mathcal{M} = \binom{[n]}{k}$ and Muller–Speyer [29] in general.

Theorem 2.4. *For any reduced plabic graph Γ with $\mathcal{M}_{\Gamma} = \mathcal{M}$, the map*

$$\mathbf{F}_{\Gamma} : \Pi_{\mathcal{M}}^{>0} \xrightarrow{\sim} \mathcal{A}_{\Gamma}^{>0}$$

is a bijection. Moreover, $\mathbf{F}_{\Gamma}$ is compatible with moves in the following sense. For any move $\Gamma \rightarrow \Gamma'$, the diagram

$$(5) \quad \begin{array}{ccc} \mathcal{A}_{\Gamma}^{>0} & \xrightarrow{\sim} & \mathcal{A}_{\Gamma'}^{>0} \\ \mathbf{F}_{\Gamma} \swarrow & & \searrow \mathbf{F}_{\Gamma'} \\ & \Pi_{\mathcal{M}}^{>0} & \end{array}.$$

commutes. Therefore, the maps $(\mathbf{F}_{\Gamma})_{\Gamma: \mathcal{M}_{\Gamma} = \mathcal{M}}$ glue together to a bijection

$$\mathbf{F}_{\mathcal{M}} : \Pi_{\mathcal{M}}^{>0} \xrightarrow{\sim} \mathcal{A}_{\mathcal{M}}^{>0}.$$

The target face labels change under moves as shown in Figure 5. Therefore, the commutativity of (5) is trivial for the contraction-uncontraction move and is a consequence of the three-term Plücker relation

$$\Delta_{Sik} \Delta_{Sjl} = \Delta_{Sij} \Delta_{Skl} + \Delta_{Sil} \Delta_{Sjk}$$

for the spider move.

The group $\mathbb{R}_{>0}^n$ acts on $\mathcal{A}_\Gamma^{>0}$ by

$$\lambda \cdot A = \left(\left(\prod_{i \in \mathbf{T}(f)} \lambda_i \right) A_f \right)_{f \in F(\Gamma)}$$

making the map $\mathbf{F}_\Gamma$ equivariant. The action is compatible with moves, and therefore the same is true for $\mathbf{F}_\mathcal{M}$.

2.5. The twist maps. Let $\mathcal{M}$ be a positroid, let $\mathbf{I}_\mathcal{M} = (I_1, \dots, I_n)$ be its Grassmann necklace, and let $\Pi_\mathcal{M}^{>0}$ denote the corresponding positroid cell. The *right twist* $\vec{\tau}_\mathcal{M}$ is the map

$$\vec{\tau}_\mathcal{M} : \Pi_\mathcal{M}^{>0} \rightarrow \Pi_\mathcal{M}^{>0}$$

defined as follows. Let M be a $k \times n$ matrix representative of $V \in \Pi_\mathcal{M}$, and denote by M_i the i -th column of M . Define a new matrix $\vec{\tau}_\mathcal{M}(M)$ by the conditions

$$(6) \quad \langle \vec{\tau}_\mathcal{M}(M)_i, M_i \rangle = 1, \quad \langle \vec{\tau}_\mathcal{M}(M)_i, M_j \rangle = 0 \quad \text{for } j \in I_i \setminus \{i\}.$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}^k$. Finally, set

$$\vec{\tau}_\mathcal{M}(V) := \text{rowspan}(\vec{\tau}_\mathcal{M}(M)).$$

The *left twist* $\overleftarrow{\tau}_\mathcal{M}$ is defined similarly but using the reverse Grassmann necklace instead in (6).

Lemma 2.5 (Muller–Speyer [29]). *The right twist has the following properties:*

- (1) For any $\lambda \in \mathbb{R}_{>0}^n$, $\vec{\tau}_\mathcal{M}(\lambda \cdot V) = \lambda^{-1} \cdot \vec{\tau}_\mathcal{M}(V)$.
- (2) For any boundary face $f_{i,i+1}^\partial$, $\Delta_{\mathbf{T}(f_{i,i+1}^\partial)}(\vec{\tau}_\mathcal{M}(V)) = \Delta_{\mathbf{T}(f_{i,i+1}^\partial)}(V)^{-1}$.

Let $\mathcal{M}$ be a positroid. There is a canonical map

$$p_\mathcal{M} : \mathcal{A}_\mathcal{M}^{>0} \rightarrow \mathcal{X}_\mathcal{M}^{>0},$$

called the *cluster ensemble map*, defined as follows. Let Γ be a reduced plabic graph such that $\mathcal{M}_\Gamma = \mathcal{M}$. Let $A \in \mathcal{A}_\Gamma^{>0}$, let

$$\text{wt}(e) := \begin{cases} 1/A_f A_g, & \text{if } e \text{ is internal and adjacent to faces } f \text{ and } g, \\ 1/A_{f_{i-1,i}^\partial}, & \text{if } e \text{ is adjacent to } u_i^\partial \text{ and a white internal vertex,} \\ 1/A_{f_{i,i+1}^\partial}, & \text{if } e \text{ is adjacent to } u_i^\partial \text{ and a black internal vertex,} \end{cases}$$

and define $p_\Gamma(A) := [\text{wt}]$. If $\Gamma \rightarrow \Gamma'$ is a move, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_\Gamma^{>0} & \xrightarrow{\sim} & \mathcal{A}_{\Gamma'}^{>0} \\ p_\Gamma \downarrow & & \downarrow p_{\Gamma'} \\ \mathcal{X}_\Gamma^{>0} & \xrightarrow{\sim} & \mathcal{X}_{\Gamma'}^{>0} \end{array}$$

Thus, the maps $(p_\Gamma)_{\Gamma: \mathcal{M}_\Gamma = \mathcal{M}}$ glue together to give $p_\mathcal{M}$.

Theorem 2.6 (Muller–Speyer [29]). *The right twist is a bijection whose inverse is the left twist. They fit into the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{M}}^{>0} & \xrightarrow{p_{\mathcal{M}}} & \mathcal{X}_{\mathcal{M}}^{>0} \\ \mathbf{F}_{\mathcal{M}} \uparrow & & \downarrow \text{Meas}_{\mathcal{M}} \\ \Pi_{\mathcal{M}}^{>0} & \xleftarrow{\tilde{\tau}_{\mathcal{M}}} & \Pi_{\mathcal{M}}^{>0} \\ & \xrightarrow{\tilde{\tau}_{\mathcal{M}}} & \end{array}$$

In particular, the boundary measurement map can be inverted as

$$\text{Meas}_{\mathcal{M}}^{-1} = p_{\mathcal{M}} \circ \mathbf{F}_{\mathcal{M}} \circ \tilde{\tau}_{\mathcal{M}}.$$

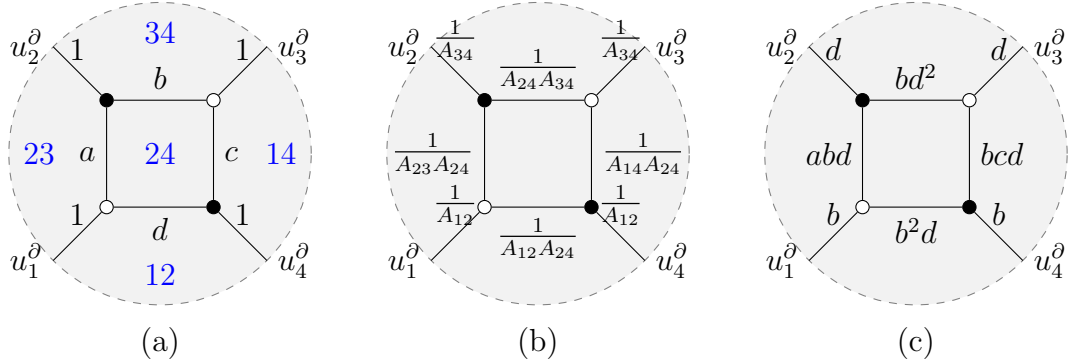


FIGURE 6. Inverting boundary measurement via the left twist after cyclically rotating the boundary labels.

Example 2.7. Consider the reduced plabic graph Γ in Figure 6(a). Its boundary measurement is

$$\Delta_{12} = b, \quad \Delta_{13} = 1, \quad \Delta_{14} = c, \quad \Delta_{23} = a, \quad \Delta_{24} = ac + bd, \quad \Delta_{34} = d.$$

Since all Plücker coordinates are positive, the positroid $\mathcal{M} = \binom{[4]}{2}$ and the positroid cell $\Pi_{\mathcal{M}}^{>0} = \text{Gr}_{2,4}^{>0}$. A matrix representative is given by

$$M = \begin{bmatrix} 1 & a & 0 & -d \\ 0 & b & 1 & c \end{bmatrix}.$$

The reverse Grassmann necklace is

$$\mathbf{I}_{\Gamma}^{\text{rev}} = (14, 12, 23, 34),$$

using which the left twist is computed to be

$$\tilde{\tau}_{\mathcal{M}}(M) = \begin{bmatrix} 1 & 0 & -\frac{b}{a} & -\frac{1}{d} \\ \frac{d}{c} & \frac{1}{b} & 1 & 0 \end{bmatrix}.$$

The twisted Plücker coordinates are given by

$$(7) \quad A_{12} = \frac{1}{b}, \quad A_{13} = \frac{ac + bd}{ac}, \quad A_{14} = \frac{1}{c}, \quad A_{23} = \frac{1}{a}, \quad A_{24} = \frac{1}{bd}, \quad A_{34} = \frac{1}{d}.$$

The target face labels are shown in Figure 6(a), using which the map p_{Γ} is computed as in Figure 6(b). Plugging in (7), we get the edge weights shown in Figure 6(c), which are gauge equivalent to the edge weights in Figure 6(a).

3. THE POSITIVE ORTHOGONAL GRASSMANNIAN AND THE ISING MODEL

Let $n \geq 1$ be a positive integer. We consider the following quadratic form on $\mathbb{R}^{2n}$

$$q: \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad \mathbf{x} = (x_1, \dots, x_{2n}) \mapsto \sum_{i=1}^{2n} (-1)^{i-1} x_i^2.$$

We denote by $Q := \text{diag}(1, -1, 1, \dots, 1, -1) \in \mathbb{R}^{2n \times 2n}$ the matrix that represents q in the standard basis. The *orthogonal Grassmannian* $\text{OGr}_{n,2n}$ is the subvariety of the Grassmannian $\text{Gr}_{n,2n}$ that consists of vector spaces V such that

$$q(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in V.$$

It is known that this variety has two isomorphic connected components that are irreducible and cut out in $\text{Gr}_{n,2n}$ by linear equations as follows

$$\text{OGr}_{n,2n}^{\pm} := \left\{ V \in \text{Gr}_{n,2n} : \Delta_I(V) = \pm \Delta_{I^c}(V) \quad \text{for all } I \in \binom{[2n]}{n} \right\}.$$

We define

$$\text{OGr}_{n,2n}^{\geq 0} := \text{OGr}_{n,2n} \cap \text{Gr}_{n,2n}^{\geq 0} \quad \text{and} \quad \text{OGr}_{n,2n}^{> 0} := \text{OGr}_{n,2n} \cap \text{Gr}_{n,2n}^{> 0}.$$

Note that $\text{OGr}_{n,2n}^{\geq 0} \subset \text{OGr}_{n,2n}^{+}$. Intersecting $\text{OGr}_{n,2n}^{\geq 0}$ with the positroid stratification (4), we get an induced stratification

$$\text{OGr}_{n,2n}^{\geq 0} = \bigsqcup_{\mathcal{M}} \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}}^{> 0}.$$

Let us denote by P_n the set of fixed-point-free involutions on $[2n]$. Given $\pi \in P_n$, viewing it as a decorated permutation, we have an associated positroid $\mathcal{M}_{\pi}$ via the bijections in Theorem 2.2.

Theorem 3.1 (Galashin–Pylyavskyy [12]). *The positroid cells that have nonempty intersection with $\text{OGr}_{n,2n}^{\geq 0}$ are precisely those of the form $\Pi_{\mathcal{M}_{\pi}}^{> 0}$ for a fixed-point-free involution $\pi : [2n] \rightarrow [2n]$. Thus,*

$$\text{OGr}_{n,2n}^{\geq 0} = \bigsqcup_{\pi \in P_n} (\text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_{\pi}}^{> 0}).$$

3.1. Cactus graphs. Let $\mathbb{D}$ be a disk with boundary vertices labeled $b_1^{\partial}, \dots, b_n^{\partial}$ in clockwise cyclic order. Let P be a noncrossing partition of $[n]$. A *cactus with shape P* is the topological space $\mathbb{D}/P$ obtained by gluing together the points $b_1^{\partial}, \dots, b_n^{\partial}$ whose indices are in the same part of P . A *cactus graph with shape P* is a graph G embedded in a cactus $\mathbb{D}/P$ with boundary vertices $b_1^{\partial}, \dots, b_n^{\partial}$.

Let G be a cactus graph with shape P . The *medial graph* $G^{\times}$ of G is the graph in $\mathbb{D}$ obtained from G as follows. Place boundary vertices $u_1^{\partial}, \dots, u_{2n}^{\partial}$ of $G^{\times}$ around the boundary of the cactus $\mathbb{D}/P$ so that b_i^{∂} is between u_{2i-1}^{∂} and u_{2i}^{∂} . Replace each edge of G as in Figure 7. Finally, take the preimage of the resulting graph under the quotient map $\mathbb{D} \rightarrow \mathbb{D}/P$. By construction, each internal vertex of $G^{\times}$ has degree 4. A *strand* of G is a maximal (unoriented) path in $G^{\times}$ that contains the opposite pair of edges at each internal vertex. The *medial pairing* of G is the fixed-point-free involution $\pi_G : [2n] \rightarrow [2n]$ given by

$$\pi_G(i) = j \quad \text{if there is a strand between } u_i^{\partial} \text{ and } u_j^{\partial}.$$

If $\pi_G(i) = j$, then we write $j = \bar{i}$. We denote the strand between i and $\bar{i}$ by $\gamma_{\{i, \bar{i}\}}$.

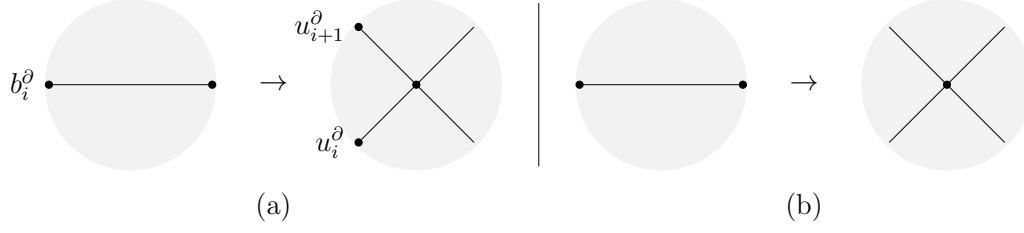


FIGURE 7. The construction of the medial graph $G^\times$ from G at (a) boundary edges and (b) internal edges.

We say that a cactus graph G is *reduced* if:

- (1) No strand is a closed loop.
- (2) No pair of strands intersects twice.

The *Y- Δ move* is a local modification of cactus graphs shown in Figure 8. We say that two cactus graphs G and G' are *move-equivalent* if they are connected by a sequence of Y- Δ moves. Similarly to Theorem 2.2, we have:

Proposition 3.2. *The map*

$$\left\{ \text{reduced cactus graphs} \right\} / \text{move-equivalence} \xrightarrow{G \mapsto \pi_G} \left\{ \text{fixed-point-free involutions of } [2n] \right\}$$

is well-defined and a bijection.

In other words, move-equivalence classes of reduced cactus graphs with n boundary vertices are classified by fixed-point-free involutions of $[2n]$.

3.2. Ising models. An *Ising model* is a pair (G, J) where G is a reduced cactus graph and $J : E(G) \rightarrow \mathbb{R}_{>0}$ is a function called *coupling constant*. Let

$$\mathcal{I}_G^{>0} \cong \mathbb{R}_{>0}^{E(G)}$$

be the space of Ising models with underlying graph G . Each Y- Δ move $G \rightarrow G'$ induces a bijection between the corresponding spaces of Ising models

$$\mathcal{I}_G^{>0} \xrightarrow{\sim} \mathcal{I}_{G'}^{>0}$$

as illustrated in Figure 8. The edge parameters shown in the figure are $\exp(2J_e)$ and the bijection is given in terms of these parameters by

$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}}, \quad B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}}, \quad C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}.$$

Let π be a fixed-point-free involution on $[2n]$. The *space of Ising models associated to π* is defined as

$$\mathcal{I}_\pi^{>0} := \bigsqcup_{G: \pi_G = \pi} \mathcal{I}_G^{>0} / \text{Y-}\Delta \text{ moves.}$$

A *spin configuration* is a function $\sigma : V \rightarrow \{-1, 1\}$ assigning to each vertex $v \in V$ a spin σ_v . The probability of a spin configuration is defined as

$$\mathbf{P}(\sigma) := \frac{1}{Z} \prod_{e=uv \in E} e^{J_e \sigma_u \sigma_v},$$

where $Z := \sum_{\sigma: V \rightarrow \{-1,1\}} \prod_{e=uv \in E} e^{J_e \sigma_u \sigma_v}$ is the *partition function*. Given $i, j \in [n]$, we define the *boundary correlation function* by

$$\langle \sigma_i \sigma_j \rangle := \sum_{\sigma: V \rightarrow \{-1,1\}} \mathbf{P}(\sigma) \sigma_{b_i^\partial} \sigma_{b_j^\partial}.$$

By definition, we have $\langle \sigma_i \sigma_j \rangle = \langle \sigma_j \sigma_i \rangle$ and $\langle \sigma_i \sigma_i \rangle = 1$ for all $i, j \in [n]$. Thus, the $n \times n$ -matrix

$$M(G, J) := (\langle \sigma_i \sigma_j \rangle)_{i,j \in [n]}$$

of boundary correlation functions is a symmetric real matrix with 1s on the diagonal. We denote the space of such matrices by $\text{Mat}_n^{\text{sym}}(\mathbb{R}, 1)$. The map

$$\text{Corr}_G : \mathcal{I}_G^{>0} \rightarrow \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1), \quad J \mapsto M(G, J)$$

is called the *boundary correlation map*.

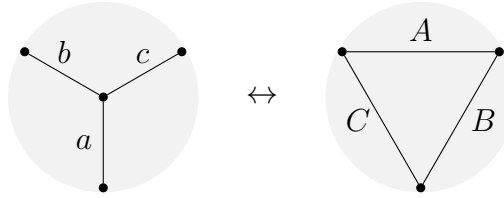


FIGURE 8. The Ising Y- Δ move.

Let

$$\phi_n : \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1) \rightarrow \text{Gr}_{n,2n}$$

be the *doubling map* defined as follows: given $M = (m_{i,j}) \in \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1)$, let $\phi_n(M) = \text{rowspan}(\tilde{M})$ where $\tilde{M} = (\tilde{M}_{i,j})$ is the $n \times 2n$ -matrix defined as follows. For $i = j$, let

$$\tilde{M}_{i,2i-1} = \tilde{M}_{i,2i} = m_{i,i} = 1,$$

and for $i \neq j$, let

$$\tilde{M}_{i,2j-1} = -\tilde{M}_{i,2j} = (-1)^{i+j+\mathbb{1}(i < j)} m_{i,j},$$

where $\mathbb{1}(i < j)$ denotes 1 if $i < j$ and 0 otherwise. Similarly to Theorem 2.3, we have:

Theorem 3.3 (Galashin–Pylyavskyy [12]). *Let G be a reduced cactus graph and let $\pi = \pi_G$ be its fixed-point-free involution. Then, the composition*

$$\mathcal{I}_G^{>0} \xrightarrow{\text{Corr}_G} \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1) \xrightarrow{\phi_n} \text{Gr}_{n,2n}^{\geq 0}$$

is a bijection

$$\mathcal{I}_G^{>0} \xrightarrow{\sim} \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}.$$

The boundary correlation map is compatible with Y- Δ moves, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}_G^{>0} & \xrightarrow{\sim} & \mathcal{I}_{G'}^{>0} \\ \text{Corr}_G \searrow & & \swarrow \text{Corr}_{G'} \\ & \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1) & \end{array}.$$

Therefore, the maps $(\text{Corr}_G)_{G:\pi_G=\pi}$ glue together to a map

$$\text{Corr}_\pi : \mathcal{I}_\pi^{>0} \rightarrow \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1).$$

which gives a bijection

$$\phi_n \circ \text{Corr}_\pi : \mathcal{I}_\pi^{>0} \xrightarrow{\sim} \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}.$$

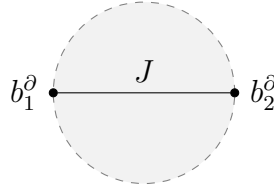


FIGURE 9. An Ising model with $n = 2$.

Example 3.4. Consider the Ising model shown in Figure 9. Then

$$M = \begin{bmatrix} 1 & \langle \sigma_1 \sigma_2 \rangle \\ \langle \sigma_1 \sigma_2 \rangle & 1 \end{bmatrix}, \quad \text{where } \langle \sigma_1 \sigma_2 \rangle = \frac{e^J - e^{-J}}{e^J + e^{-J}} = \tanh J.$$

The point in $\text{OGr}_{2,4}^{\geq 0}$ is the rowspan of

$$\tilde{M} = \phi_2(M) = \begin{bmatrix} 1 & 1 & \langle \sigma_1 \sigma_2 \rangle & -\langle \sigma_1 \sigma_2 \rangle \\ -\langle \sigma_1 \sigma_2 \rangle & \langle \sigma_1 \sigma_2 \rangle & 1 & 1 \end{bmatrix}$$

with Plücker coordinates

$$(8) \quad \Delta_{12} = \Delta_{34} = 2\langle \sigma_1 \sigma_2 \rangle, \quad \Delta_{14} = \Delta_{23} = 1 - \langle \sigma_1 \sigma_2 \rangle^2, \quad \Delta_{13} = \Delta_{24} = 1 + \langle \sigma_1 \sigma_2 \rangle^2.$$

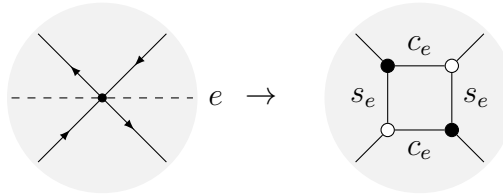


FIGURE 10. The construction of the dimer model $(G^\square, [\text{wt}^\square])$ from $G^\times$ and J . The edges of $G^\square$ with no edge weight indicated have edge weight 1.

3.3. Embedding Ising models in dimer models. Let G be a reduced cactus graph. We orient the edges of the medial graph $G^\times$ so that:

- (1) The boundary vertex u_i^∂ is a source if and only if i is odd.
- (2) Each internal vertex of $G^\times$ is incident to two incoming and two outgoing edges which alternate in orientation around the vertex.

The existence of this ordering shows that π_G pairs even numbers with odd numbers and vice versa. We construct a reduced plabic graph $G^\square$ by replacing each vertex of $G^\times$ as shown on the left hand side of Figure 10 with the bipartite graph on the right hand side. We refer to the degree-4 faces of $G^\square$ corresponding to edges of G as *square faces*. The remaining faces of $G^\square$ are in bijection with the vertices and faces of G . Note that there is a 2 : 1 correspondence between strands in $G^\square$ and strands in G as follows. The strand in G with source u_i^∂ corresponds to the pair of strands in $G^\square$ with sources $u_i^\partial, u_i^\partial$. Moreover, we have $\pi_{G^\square} = \pi_G$.

Given an Ising model (G, J) , we define the two functions $s, c : E(G) \rightarrow (0, 1)$ by

$$s_e := \text{sech}(2J_e), \quad c_e := \tanh(2J_e)$$

for every edge $e \in E(G)$ and replace e with the dimer model $(G^\square, [\text{wt}^\square])$ shown in Figure 10. The map

$$\mathcal{I}_G^{>0} \hookrightarrow \mathcal{X}_{G^\square}^{>0}, \quad (G, J) \mapsto (G^\square, [\text{wt}^\square]).$$

is injective (see the proof of Theorem 5.17 in [12]).

Proposition 3.5 (Kenyon–Pemantle [22]). *Let $G \rightarrow G'$ be a Y - Δ move. Then there is a sequence of moves $G^\square \rightarrow (G')^\square$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{I}_G^{>0} & \xrightarrow{\sim} & \mathcal{I}_{G'}^{>0} \\ \downarrow & & \downarrow \\ \mathcal{X}_{G^\square}^{>0} & \xrightarrow{\sim} & \mathcal{X}_{(G')^\square}^{>0} \end{array}.$$

In particular, the embeddings $\mathcal{I}_G^{>0} \hookrightarrow \mathcal{X}_{G^\square}^{>0}$ glue together to give an embedding

$$\mathcal{I}_\pi^{>0} \hookrightarrow \mathcal{X}_{\mathcal{M}_\pi}^{>0}.$$

Theorem 3.6 (Galashin–Pylyavskyy [12]). *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{I}_\pi^{>0} & \hookrightarrow & \mathcal{X}_{\mathcal{M}_\pi}^{>0} \\ \text{Corr}_\pi \downarrow & & \text{Meas}_{\mathcal{M}_\pi} \downarrow \\ & & \Pi_{\mathcal{M}_\pi}^{>0} \\ & \uparrow & \\ \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1) & \xrightarrow{\phi_n} & \text{OGr}_{n, 2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0} \end{array}$$

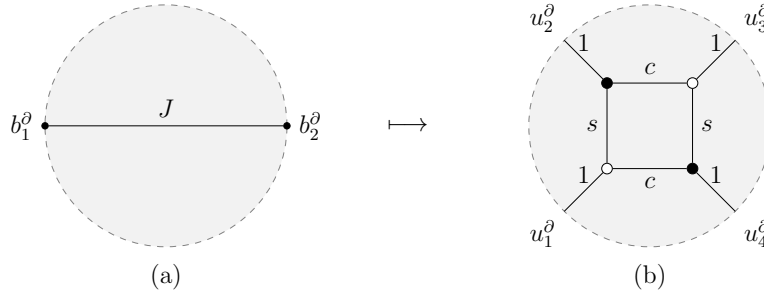


FIGURE 11. The Ising model from Figure 9 and the corresponding dimer model. Here $c = \tanh(2J)$ and $s = \text{sech}(2J)$.

Example 3.7. Consider the Ising model in Figure 11(a). The associated point in $\text{OGr}_{2,4}^{>0}$ was computed in Example 3.4. Since Plücker coordinates are projective, we may divide (8) by $1 + \langle \sigma_1 \sigma_2 \rangle^2$ and write the same point as

$$\Delta_{12} = \Delta_{34} = \frac{2\langle \sigma_1 \sigma_2 \rangle}{1 + \langle \sigma_1 \sigma_2 \rangle^2}, \quad \Delta_{13} = \Delta_{24} = 1, \quad \Delta_{14} = \Delta_{23} = \frac{1 - \langle \sigma_1 \sigma_2 \rangle^2}{1 + \langle \sigma_1 \sigma_2 \rangle^2}.$$

Using the identities

$$\frac{2 \tanh J}{1 + \tanh^2 J} = \tanh(2J), \quad \frac{1 - \tanh^2 J}{1 + \tanh^2 J} = \text{sech}(2J),$$

and setting

$$c := \tanh(2J), \quad s := \operatorname{sech}(2J),$$

we obtain

$$(9) \quad \Delta_{12} = c, \quad \Delta_{13} = 1, \quad \Delta_{14} = s, \quad \Delta_{23} = s, \quad \Delta_{24} = 1, \quad \Delta_{34} = c.$$

On the other hand, the embedding $\mathcal{I}_\pi^{>0} \hookrightarrow \mathcal{X}_{\mathcal{M}_\pi}^{>0}$ sends the Ising model in Figure 11(a) to the dimer model in Figure 11(b), where the edge weights are

$$a = s, \quad b = c, \quad c = s, \quad d = c$$

in the notation of Figure 6. Applying the boundary measurement computed in Figure 6 and using

$$s^2 + c^2 = \operatorname{sech}^2(2J) + \tanh^2(2J) = 1,$$

we get

$$\Delta_{12} = c, \quad \Delta_{13} = 1, \quad \Delta_{14} = s, \quad \Delta_{23} = s, \quad \Delta_{24} = s^2 + c^2 = 1, \quad \Delta_{34} = c,$$

which is the same as (9), verifying Theorem 3.6 for this example.

We give another characterization of the image

$$\mathcal{I}_G^{>0} \hookrightarrow \mathcal{X}_{G^\square}^{>0}$$

found in [15] for Ising models on a torus. The proof given there however works for arbitrary surfaces. Let $\bar{G}^\square$ be the plabic graph obtained from $G^\square$ by changing the colors of all the vertices. We define two operations:

- (1) Color-change: We replace $G^\square$ with $\bar{G}^\square$, i.e., change colors of all vertices, but keep all the edge weights the same.
- (2) Spider moves at all square faces: We perform a spider move at each square face of $G^\square$.

Proposition 3.8 (George [15]). *[wt] $\in \mathcal{X}_{G^\square}^{>0}$ is in the image of $\mathcal{I}_G^{>0}$ if and only if color-change results in the same gauge-equivalence class of edge weights as spider moves at all square faces.*

Under boundary measurement, color-change is the map $\Delta_I \mapsto \Delta_{[2n] \setminus I}$ while doing spider moves at all square faces does not change the boundary measurement. Since $\operatorname{OGr}_{n,2n}^+$ is defined by $\Delta_I = \Delta_{[2n] \setminus I}$ for all $I \in \binom{[2n]}{n}$, this gives an alternative proof that the composition

$$\mathcal{I}_\pi^{>0} \hookrightarrow \mathcal{X}_{\mathcal{M}_\pi}^{>0} \xrightarrow{\operatorname{Meas}_{\mathcal{M}_\pi}} \Pi_{\mathcal{M}_\pi}^{>0}$$

lands in $\operatorname{OGr}_{n,2n}^{>0}$. We will see that an identical characterization exists on the cluster $\mathcal{A}$ side (Remark 4.1). However, the cluster ensemble and twist maps are not compatible with color-change and will need to be modified (see Section 5).

Example 3.9. Consider the dimer model $(G^\square, [\text{wt}])$ shown in Figure 12(a). The results of color-change and spider moves at all square faces are shown in (b) and (c) respectively. The characterization in Proposition 3.8 says that the image of $\mathcal{I}_G^{>0}$ is defined by

$$a = \frac{c}{ac + bd}, \quad b = \frac{d}{ac + bd}, \quad c = \frac{a}{ac + bd}, \quad d = \frac{b}{ac + bd}.$$

The first and third equations imply that $a^2 = c^2$. Since $a, c > 0$, this gives $a = c$. Similarly, the second and the fourth equations imply that $b = d$. Finally, any of the four equations gives $a^2 + b^2 = 1$, so we get $s_e = a, c_e = b$.

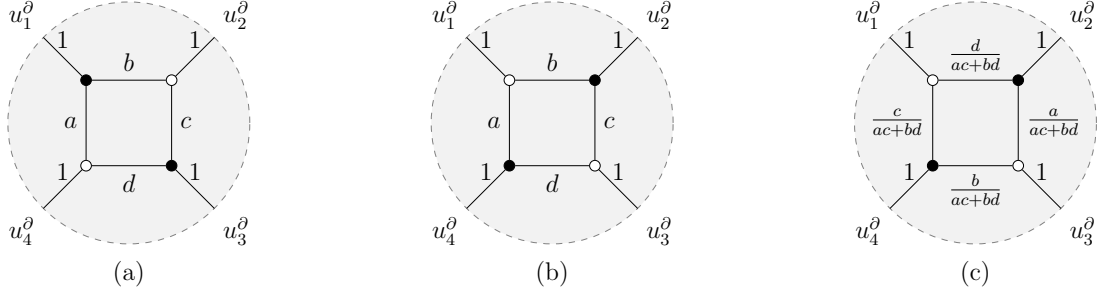


FIGURE 12. A plabic-graph $G^\square$ (a), the result of color-change (b) and spider moves at all square faces (c).

4. C -NETWORKS AND LOCALLY LAGRANGIAN POSITROID CELLS

4.1. **C -networks.** A C -network is a pair (G, C) where G is a reduced cactus graph and

$$C : V(G) \sqcup F(G) \rightarrow \mathbb{R}_{>0}$$

is a function defined modulo rescaling by $\mathbb{R}_{>0}$. Let

$$\mathcal{C}_G^{>0} \cong \mathbb{R}_{>0}^{|V(G)|+|F(G)|-1}$$

be the *space of C -networks with underlying cactus graph G* .

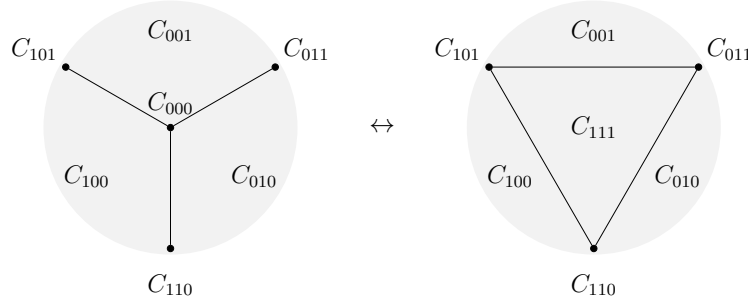


FIGURE 13. The Y- Δ move for C -networks.

A Y- Δ move $G \rightarrow G'$ induces a map

$$\mathcal{C}_G^{>0} \xrightarrow{\sim} \mathcal{C}_{G'}^{>0}$$

given in the notation of Figure 13 by the *positive Kashaev equation*

$$C_{111} = \frac{1}{C_{000}^2} \left(2C_{100}C_{010}C_{001} + C_{000}(C_{100}C_{011} + C_{010}C_{101} + C_{001}C_{110}) \right. \\ \left. + 2\sqrt{(C_{000}C_{011} + C_{010}C_{001})(C_{000}C_{101} + C_{100}C_{001})(C_{000}C_{110} + C_{100}C_{010})} \right)$$

By construction, we have a bijection

$$V(G) \sqcup E(G) \sqcup F(G) \xrightarrow{\sim} F(G^\square).$$

Note that the edges of G correspond to the square faces of $G^\square$. We define an embedding

$$\mathcal{C}_G^{>0} \hookrightarrow \mathcal{A}_{G^\square}^{>0}$$

as follows. We let $A_x := C_x$ for $x \in V(G) \sqcup F(G)$ and for an edge $e = uv \in E(G)$ incident to faces f, g , we let

$$(10) \quad A_e := \sqrt{C_u C_v + C_f C_g}.$$

Remark 4.1. We can interpret (10) as saying that color-change gives the same A -network as applying spider moves at all square faces:

- (1) Color-change: We replace $G^\square$ with $\overline{G^\square}$, i.e., change colors of all vertices, but keep the face weights the same.
- (2) Spider moves at all square faces: We perform a spider move at each square face of $G^\square$, which replaces A_e by $(A_u A_v + A_f A_g)/A_e$.

Setting $(A_u A_v + A_f A_g)/A_e = A_e$, we get (10).

Proposition 4.2 (Kenyon–Pemantle [22]). *Let $G \rightarrow G'$ be a Y - Δ move. Then there is a sequence of moves $G^\square \rightarrow (G')^\square$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}_G^{>0} & \xrightarrow{\sim} & \mathcal{C}_{G'}^{>0} \\ \downarrow & & \downarrow \\ \mathcal{A}_{G^\square}^{>0} & \xrightarrow{\sim} & \mathcal{A}_{(G')^\square}^{>0} \end{array}.$$

In particular, the embeddings $\mathcal{C}_G^{>0} \hookrightarrow \mathcal{A}_{G^\square}^{>0}$ glue to give an embedding

$$\mathcal{C}_\pi^{>0} \hookrightarrow \mathcal{A}_{\mathcal{M}_\pi}^{>0},$$

where $\pi = \pi_G = \pi_{G'}$.

We now define the subset of $\Pi_{\mathcal{M}_\pi}$ parameterized by C -networks. We denote by $\mathcal{S}_G^\square$ the set of all $I \in \binom{[2n]}{n}$ which appear as the target label of a square face of $G^\square$, where G is a reduced cactus graph with $\pi_G = \pi$, and let $\mathcal{S}_\pi^\square := \bigcup_{G: \pi_G = \pi} \mathcal{S}_G^\square$.

Any such I is of the form $S\bar{i}\bar{i}$ where:

- (1) $\{\gamma_{\{i, \bar{i}\}}, \gamma_{\{j, \bar{j}\}}\}$ is a *crossing*, i.e., a pair of strands $\{\gamma_{\{i, \bar{i}\}}, \gamma_{\{j, \bar{j}\}}\}$ with $i < j < \bar{i} < \bar{j}$ in cyclic order around the boundary of $\mathbb{D}$.
- (2) S is an $(n-2)$ -element subset of $[2n]$ that contains exactly one element from each pair $\{k, \bar{k}\}$ for all strands $\gamma_{\{k, \bar{k}\}} \neq \gamma_{\{i, \bar{i}\}}, \gamma_{\{j, \bar{j}\}}$.

Consider the condition

$$(11) \quad \Delta_{S\bar{i}\bar{i}}(V) = \Delta_{Sj\bar{j}}(V).$$

Definition 4.3 (Locally Lagrangian positroid cells). For any matching $\pi \in P_n$, we define the *locally Lagrangian positroid cell* $\Lambda_\pi^{>0} \subset \Pi_{\mathcal{M}_\pi}^{>0}$ as the subset of subspaces V that satisfy (11) for every $I \in \mathcal{S}_\pi^\square$.

When π is the matching $i \mapsto i + n$ (modulo $2n$), $\Lambda_\pi^{>0}$ is the intersection of the Lagrangian Grassmannian with $\text{Gr}_{n, 2n}^{>0}$ or equivalently the space of positive-definite Hermitian matrices [21, Section 5]. In general, (11) can be viewed as a local Lagrangian condition, but we do not know a global description of these spaces.

Proposition 4.4. *Let π be a matching on $[2n]$ and let $V \in \Pi_{\mathcal{M}_\pi}^{>0}$. Then $V \in \Lambda_\pi^{>0}$ if and only if the following condition holds:*

For every square-face label $I = Si\bar{i} \in \mathcal{S}_\pi^\square$, let $Sj\bar{j}$ be the label obtained from the same local square after applying the spider move such that $i < j < \bar{i} < \bar{j}$. Set $W_S := \text{span}\{e_s : s \in [2n] \setminus S\} \subset \mathbb{R}^{2n}$, $U_{ij} := \text{span}\{e_i, e_{\bar{i}}, e_j, e_{\bar{j}}\}$, and let

$$\text{pr}_{ij} : \mathbb{R}^{2n} \rightarrow U_{ij}$$

be the coordinate projection. We equip the space U_{ij} with the symplectic form

$$\omega_{ij} := e_i^* \wedge e_{\bar{i}}^* - \epsilon_{Sij} e_j^* \wedge e_{\bar{j}}^*, \quad \text{where} \quad \epsilon_{Sij} := (-1)^{|S \cap (i,j)| + |S \cap (\bar{i},\bar{j})|}.$$

Here (i, j) is the open interval $(i, j) := \{i+1, \dots, j-1\}$. Then the vector space $\text{pr}_{ij}(V \cap W_S) \subset U_{ij}$ is a Lagrangian 2-plane with respect to ω_{ij} .

Proof. Let M be an $n \times 2n$ matrix whose rowspan is V . Let M' be the matrix obtained from M by permuting the columns so that the columns corresponding to S appear as the first $n-2$ columns. Then,

$$(12) \quad \Delta_{Si\bar{i}}(M') = (-1)^{|S \cap (i,\bar{i})|} \Delta_{Si\bar{i}}(M), \quad \Delta_{Sj\bar{j}}(M') = (-1)^{|S \cap (j,\bar{j})|} \Delta_{Sj\bar{j}}(M).$$

Since $\Delta_{Si\bar{i}}(V) \neq 0$, by performing row operations, we may assume that the columns in $Si\bar{i}$ contain the identity matrix. Then $V \cap W_S$ is the row span of the matrix obtained from M by deleting the rows corresponding to S in the identity matrix, which leaves a $2 \times 2n$ matrix. The projection $\text{pr}_{ij}(V \cap W_S)$ is then obtained by deleting the columns other than $i, \bar{i}, j, \bar{j}$. By construction, every Plücker coordinate of $\text{pr}_{ij}(V \cap W_S)$ is equal to the Plücker coordinate of M' obtained by adjoining S . Thus $\text{pr}_{ij}(V \cap W_S)$ is Lagrangian in U_{ij} if and only if $\Delta_{Si\bar{i}}(M') = \epsilon_{Sij} \Delta_{Sj\bar{j}}(M')$. By (12), this is equivalent to (11). $\square$

Example 4.5 ($n = 3$). We consider the matching π from Figure 2. The cell $\Lambda_\pi^{>0}$ is cut out by the following conditions:

- (1) $\Delta_{126} = \Delta_{345} = 0$ and $\Delta_I > 0$ for $I \in \binom{[6]}{3} \setminus \{\{1, 2, 6\}, \{3, 4, 5\}\}$ (positroid condition),
- (2) $\Delta_{356} = \Delta_{146}$ and $\Delta_{134} = \Delta_{236}$ (square faces condition).

Definition 4.6 (The map $\mathbf{H}_G$ from locally Lagrangian cells to C -networks). Given a reduced cactus graph G with $\pi_G = \pi$, we define the map

$$\mathbf{H}_G : \Lambda_\pi^{>0} \rightarrow \mathcal{C}_G^{>0}, \quad V \mapsto (\Delta_{\mathbf{T}(x)}(V))_{x \in V(G) \sqcup F(G)},$$

where $\mathbf{T}(x)$ denotes the target face label of the corresponding face of $G^\square$.

Note that the condition (11) implies (10) and therefore the following diagram commutes:

$$(13) \quad \begin{array}{ccc} \mathcal{C}_G^{>0} & \hookrightarrow & \mathcal{A}_{G^\square}^{>0} \\ \mathbf{H}_G \uparrow & & \uparrow \mathbf{F}_{G^\square} \\ \Lambda_\pi^{>0} & \hookrightarrow & \Pi_{\mathcal{M}_\pi}^{>0} \end{array}.$$

Lemma 4.7. If $G \rightarrow G'$ is a Y - Δ move, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_G^{>0} & \xrightarrow{\sim} & \mathcal{C}_{G'}^{>0} \\ \mathbf{H}_G \swarrow & & \searrow \mathbf{H}_{G'} \\ & \Lambda_\pi^{>0} & \end{array}.$$

Proof. The proof is a simple diagram chase. By injectivity of $\mathcal{C}_{G'}^{>0} \hookrightarrow \mathcal{A}_{(G')\square}^{>0}$, it suffices to show that

$$\Lambda_\pi^{>0} \xrightarrow{H_G} \mathcal{C}_G^{>0} \xrightarrow{\sim} \mathcal{C}_{G'}^{>0} \hookrightarrow \mathcal{A}_{(G')\square}^{>0} = \Lambda_\pi^{>0} \xrightarrow{H_{G'}} \mathcal{C}_{G'}^{>0} \hookrightarrow \mathcal{A}_{(G')\square}^{>0}.$$

By Proposition 4.2, we have

$$\Lambda_\pi^{>0} \xrightarrow{H_G} \mathcal{C}_G^{>0} \xrightarrow{\sim} \mathcal{C}_{G'}^{>0} \hookrightarrow \mathcal{A}_{(G')\square}^{>0} = \Lambda_\pi^{>0} \xrightarrow{H_G} \mathcal{C}_G^{>0} \hookrightarrow \mathcal{A}_{G\square}^{>0} \xrightarrow{\sim} \mathcal{A}_{(G')\square}^{>0}.$$

By (13) and the corresponding diagram for G' , we have

$$\begin{aligned} \Lambda_\pi^{>0} \xrightarrow{H_G} \mathcal{C}_G^{>0} \hookrightarrow \mathcal{A}_{G\square}^{>0} &\xrightarrow{\sim} \mathcal{A}_{(G')\square}^{>0} = \Lambda_\pi^{>0} \hookrightarrow \Pi_{\mathcal{M}_\pi}^{>0} \xrightarrow{F_{G\square}} \mathcal{A}_{G\square}^{>0} \xrightarrow{\sim} \mathcal{A}_{(G')\square}^{>0}, \\ \Lambda_\pi^{>0} \xrightarrow{H_{G'}} \mathcal{C}_{G'}^{>0} \hookrightarrow \mathcal{A}_{(G')\square}^{>0} &= \Lambda_\pi^{>0} \hookrightarrow \Pi_{\mathcal{M}_\pi}^{>0} \xrightarrow{F_{(G')\square}} \mathcal{A}_{(G')\square}^{>0}, \end{aligned}$$

and these are equal by Theorem 2.4. $\square$

Proposition 4.8. *The map H_G is a bijection.*

Proof. H_G is injective since all the other maps in (13) are injective. For surjectivity, given $C \in \mathcal{C}_G^{>0}$, the composition $\mathcal{C}_G^{>0} \hookrightarrow \mathcal{A}_{G\square}^{>0} \xrightarrow{F_{G\square}^{-1}} \Pi_{\mathcal{M}_\pi}^{>0}$ gives us a point $V \in \Pi_{\mathcal{M}_\pi}^{>0}$ and it remains to show that $V \in \Lambda_\pi^{>0}$, i.e., to check (11). The conditions hold for $I \in \mathcal{S}_G^\square$ by construction, and they hold for $I \in \mathcal{S}_{G'}^\square$ for G' move-equivalent to G by Proposition 4.2. $\square$

Gluing the maps $(H_G)_{G:\pi_G=\pi}$, we obtain a bijection $H_\pi : \Lambda_\pi^{>0} \rightarrow \mathcal{C}_\pi^{>0}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_\pi^{>0} & \xrightarrow{\quad} & \mathcal{A}_{\mathcal{M}_\pi}^{>0} \\ H_\pi \uparrow & & \uparrow F_{\mathcal{M}_\pi} \\ \Lambda_\pi^{>0} & \xrightarrow{\quad} & \Pi_{\mathcal{M}_\pi}^{>0} \end{array}.$$

5. THE ISING CLUSTER ENSEMBLE AND TWIST MAPS

5.1. Ising cluster ensemble. We now return to the Ising model and formulate the chamber ansatz from the introduction in cluster-ensemble language.

Definition 5.1 (Ising cluster ensemble map). Let G be a reduced cactus graph. We denote by $q_G : \mathcal{C}_G^{>0} \rightarrow \mathcal{I}_G^{>0}$ the map defined as in Figure 14, where

$$(14) \quad c_e := \sqrt{\frac{C_u C_v}{C_u C_v + C_f C_g}} = \frac{\sqrt{A_u A_v}}{A_e} \quad \text{and} \quad s_e := \sqrt{\frac{C_f C_g}{C_u C_v + C_f C_g}} = \frac{\sqrt{A_f A_g}}{A_e}.$$

We call this map the *Ising cluster ensemble map*.

The following is a straightforward verification.

Proposition 5.2. *If $G \rightarrow G'$ is a Y - Δ move, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}_G^{>0} & \xrightarrow{\sim} & \mathcal{C}_{G'}^{>0} \\ q_G \downarrow & & \downarrow q_{G'} \\ \mathcal{I}_G^{>0} & \xrightarrow{\sim} & \mathcal{I}_{G'}^{>0} \end{array}.$$

Gluing the maps q_G as G varies over all reduced graphs with $\pi_G = \pi$, we get

$$q_\pi : \mathcal{C}_\pi^{>0} \rightarrow \mathcal{I}_\pi^{>0}.$$

5.2. Ising twist map. Let $\mathcal{M}$ be a positroid and let Γ be a reduced plabic graph with $\mathcal{M}_\Gamma = \mathcal{M}$. We also define a modification

$$\hat{p}_\mathcal{M} : \mathcal{A}_\mathcal{M}^{>0} \rightarrow \mathcal{X}_\mathcal{M}^{>0}.$$

of the cluster ensemble map as follows. Let $A \in \mathcal{A}_\Gamma^{>0}$ and for an edge $e = bw$ with incident faces f and g , let

$$\text{wt}(e) := \begin{cases} 1/A_f A_g, & \text{if } e \text{ is an internal edge,} \\ 1/\sqrt{A_f A_g}, & \text{if } e \text{ is a boundary edge,} \end{cases}$$

and define $\hat{p}_\Gamma(A) := [\text{wt}]$. If $\Gamma \rightarrow \Gamma'$ is a move, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_\Gamma^{>0} & \xrightarrow{\sim} & \mathcal{A}_{\Gamma'}^{>0} \\ \hat{p}_\Gamma \downarrow & & \hat{p}_{\Gamma'} \downarrow \\ \mathcal{X}_\Gamma^{>0} & \xrightarrow{\sim} & \mathcal{X}_{\Gamma'}^{>0} \end{array}.$$

Thus, the maps $(\hat{p}_\Gamma)_{\Gamma: \mathcal{M}_\Gamma = \mathcal{M}}$ glue together to give $\hat{p}_\mathcal{M}$.

Since we have modified $p_\mathcal{M}$, we also need to modify $\vec{\tau}_\mathcal{M}$ to have a commutative diagram as in Theorem 2.6.

Definition 5.3 (The Ising twist map). We define the map

$$\hat{\tau}_\mathcal{M} : \Pi_\mathcal{M}^{>0} \rightarrow \Pi_\mathcal{M}^{>0}, \quad \hat{\tau}_\mathcal{M}(V) := \left(\sqrt{\frac{\Delta_{I_1}(V)}{\Delta_{I_2}(V)}}, \dots, \sqrt{\frac{\Delta_{I_{2n}}(V)}{\Delta_{I_1}(V)}} \right) \cdot \vec{\tau}_\mathcal{M}(V)$$

where $\mathbf{I} = (I_1, \dots, I_{2n})$ is the Grassmann necklace of the positroid $\mathcal{M}$. We call this map the *Ising twist map*.

Proposition 5.4. *Let Γ be a reduced plabic graph with $\mathcal{M}_\Gamma = \mathcal{M}$. The following diagrams commute:*

$$\begin{array}{ccc} \mathcal{A}_\Gamma^{>0} & \xrightarrow{\hat{p}_\Gamma} & \mathcal{X}_\Gamma^{>0} \\ \uparrow F_\Gamma & & \downarrow \text{Meas}_\Gamma \\ \Pi_\mathcal{M}^{>0} & \xrightarrow{\hat{\tau}_\mathcal{M}} & \Pi_\mathcal{M}^{>0} \end{array} \quad \begin{array}{ccc} \mathcal{A}_\mathcal{M}^{>0} & \xrightarrow{\hat{p}_\mathcal{M}} & \mathcal{X}_\mathcal{M}^{>0} \\ \uparrow F_\mathcal{M} & & \downarrow \text{Meas}_\mathcal{M} \\ \Pi_\mathcal{M}^{>0} & \xrightarrow{\hat{\tau}_\mathcal{M}} & \Pi_\mathcal{M}^{>0} \end{array}.$$

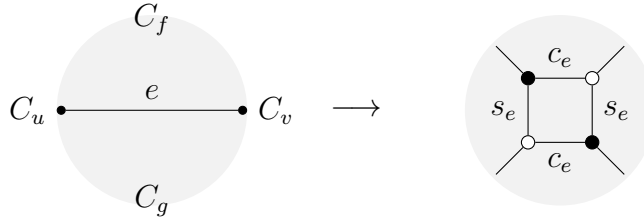


FIGURE 14. The cluster ensemble map for the Ising model. The edges with no weight indicated on the right hand side have weight 1.

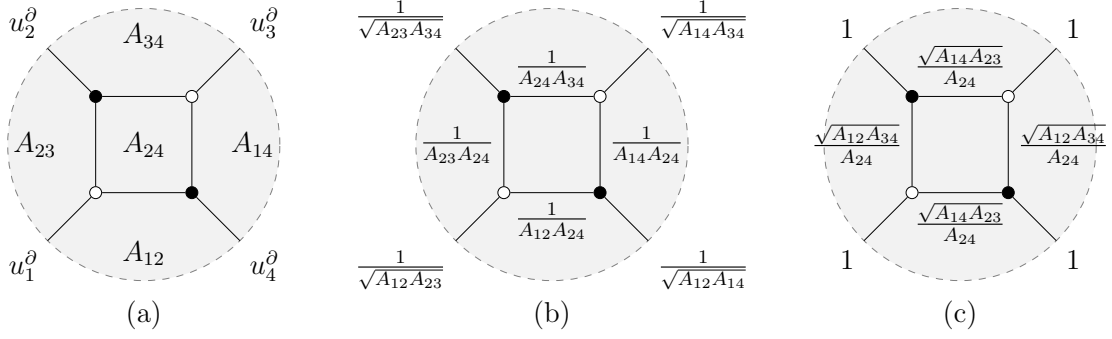


FIGURE 15. $A \in \mathcal{A}_{G^\square}^{>0}$ (a), $\hat{p}_{G^\square}(A)$ (b) and $\hat{p}_{G^\square}(A)$ after a gauge transformation (c).

Proof. We show commutativity of the first diagram by showing that $\text{Meas}_\Gamma^{-1} \circ \hat{\tau}_{\mathcal{M}} = \hat{p}_\Gamma \circ \mathbf{F}_\Gamma$; as usual, the second diagram is obtained by gluing. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{2n}) \in \mathbb{R}_{>0}^{2n}$ be given by $\lambda_i := \sqrt{\frac{\Delta_{I_i}(V)}{\Delta_{I_{i+1}}(V)}}$. We have by definition of $\hat{\tau}_{\mathcal{M}}$ and equivariance that

$$\begin{aligned} \text{Meas}_\Gamma^{-1} \circ \hat{\tau}_{\mathcal{M}}(V) &= \text{Meas}_\Gamma^{-1}(\boldsymbol{\lambda} \cdot \vec{\tau}_{\mathcal{M}}(V)) \\ &= \boldsymbol{\lambda} \cdot \text{Meas}_\Gamma^{-1}(\vec{\tau}_{\mathcal{M}}(V)) \\ &= \boldsymbol{\lambda} \cdot (p_\Gamma \circ \mathbf{F}_\Gamma \circ \vec{\tau}_{\mathcal{M}}) \circ \vec{\tau}_{\mathcal{M}}(V) \\ &= \boldsymbol{\lambda} \cdot p_\Gamma \circ \mathbf{F}_\Gamma(V). \end{aligned}$$

Therefore, it suffices to show that $\hat{p}_\Gamma \circ \mathbf{F}_\Gamma(V) = \boldsymbol{\lambda} \cdot p_\Gamma \circ \mathbf{F}_\Gamma(V)$. Clearly this is true at internal edges. We have the following two cases for boundary edges:

- (1) e is incident to u_i^∂ and a white internal vertex: $\boldsymbol{\lambda} \cdot p_\Gamma \circ \mathbf{F}_\Gamma$ assigns the weight

$$\sqrt{\frac{\Delta_{I_i}(V)}{\Delta_{I_{i+1}}(V)}} \frac{1}{\Delta_{I_i}(V)} = \frac{1}{\sqrt{\Delta_{I_i}(V)\Delta_{I_{i+1}}(V)}}.$$

- (2) e is incident to u_i^∂ and a black internal vertex: $\boldsymbol{\lambda} \cdot p_\Gamma \circ \mathbf{F}_\Gamma$ assigns the weight

$$\left(\sqrt{\frac{\Delta_{I_i}(V)}{\Delta_{I_{i+1}}(V)}} \right)^{-1} \frac{1}{\Delta_{I_{i+1}}(V)} = \frac{1}{\sqrt{\Delta_{I_i}(V)\Delta_{I_{i+1}}(V)}}. \quad \square$$

Now we return to the Ising model and relate the Ising twist map with the Ising cluster ensemble map.

Lemma 5.5. *Let G be a reduced cactus graph with $\pi_G = \pi$. The following diagrams commute:*

$$\begin{array}{ccc} \mathcal{C}_G^{>0} & \xrightarrow{q_G} & \mathcal{I}_G^{>0} \\ \downarrow & & \downarrow \\ \mathcal{A}_{G^\square}^{>0} & \xrightarrow{\hat{p}_{G^\square}} & \mathcal{X}_{G^\square}^{>0} \end{array}, \quad \begin{array}{ccc} \mathcal{C}_\pi^{>0} & \xrightarrow{q_\pi} & \mathcal{I}_\pi^{>0} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{M}_\pi}^{>0} & \xrightarrow{\hat{p}_{\mathcal{M}_\pi}} & \mathcal{X}_{\mathcal{M}_\pi}^{>0} \end{array}.$$

Proof. See Figure 15 for the proof of commutativity of the first diagram; the second is obtained by gluing. $\square$

5.3. Group action and inverse Ising twist. We have $\dim \mathcal{I}_G^{>0} = |E(G)|$ and $\dim \mathcal{C}_G^{>0} = |V(G)| + |F(G)| - 1$. Applying Euler's formula to the cell decomposition of $\mathbb{D}$ given by G together with the n intervals that $(b_i^\partial)_{i=1}^n$ divide $\partial\mathbb{D}$ into, we have $|V(G)| - (|E(G)| + n) + |F(G)| = 1$, so

$$\dim \mathcal{C}_G^{>0} - \dim \mathcal{I}_G^{>0} = n.$$

Hence, the map $\hat{\tau}_{\mathcal{M}_\pi}$ cannot be a bijection. We will see that there is an action of an n -dimensional group and $\hat{\tau}_{\mathcal{M}_\pi}$ becomes a bijection upon taking the quotient.

Given $\pi \in P_n$, consider the subgroup

$$T_\pi^{>0} := \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{2n}) : \lambda_{\bar{i}} = \lambda_i^{-1} \text{ for all } i \in [2n] \right\} \cong \mathbb{R}_{>0}^n$$

of $\mathbb{R}_{>0}^{2n}$ with the induced action.

Lemma 5.6. $T_\pi^{>0}$ stabilizes $\mathcal{C}_\pi^{>0} \subset \mathcal{A}_{\mathcal{M}_\pi}^{>0}$.

Proof. This follows from the fact that for any non-square face f of $G^\square$, the set $\mathbf{T}(f)$ contains either both i and $\bar{i}$ or neither. $\square$

Lemma 5.7. q_π is $T_\pi^{>0}$ -invariant.

Proof. Compare the target face labels corresponding to the A variables in the numerator and denominator of c and s in (14). $\square$

By Lemma 5.5 and Proposition 5.4, the induced map $q_G : \mathcal{C}_G^{>0}/T_\pi^{>0} \rightarrow \mathcal{I}_G^{>0}$ sits in the commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_G^{>0}/T_\pi^{>0} & \xrightarrow{q_G} & \mathcal{I}_G^{>0} \\ \uparrow \mathbf{H}_G & & \downarrow \phi_n \circ \text{Corr}_G \\ \Lambda_\pi^{>0}/T_\pi^{>0} & \xrightarrow{\hat{\tau}_{\mathcal{M}_\pi}} & \text{OGr}_{n,2n}^{>0} \cap \Pi_{\mathcal{M}_\pi}^{>0} \end{array}$$

and similarly, by gluing, for the induced map $q_\pi : \mathcal{C}_\pi^{>0}/T_\pi^{>0} \rightarrow \mathcal{I}_\pi^{>0}$. The goal of this section is to prove that the maps $q_G, q_\pi, \hat{\tau}_{\mathcal{M}_\pi}$ are bijections. We do this by constructing an inverse map $\hat{\tau}_{\mathcal{M}_\pi}^{-1}$ as a modification of the left twist.

Given $[\text{wt}] \in \mathcal{X}_{G^\square}$, let $m_i([\text{wt}])$ denote the alternating product of edge weights along the strand with target u_i^∂ where the edge weight $\text{wt}(bw)$ appears in the numerator if bw is traversed by the strand from b to w and in the denominator otherwise.

Lemma 5.8. Let $[\text{wt}] = \hat{p}_\Gamma(A)$ for some $A \in \mathcal{A}_\Gamma^{>0}$. Then,

$$m_i([\text{wt}]) = \sqrt{\frac{A_{f_{i,i+1}^\partial} A_{f_{\bar{i},\bar{i}+1}^\partial}}{A_{f_{i-1,i}^\partial} A_{f_{i-1,\bar{i}}^\partial}}}.$$

Proof. Suppose the edges in order along the strand with target u_i^∂ are $e_1, \dots, e_r$ and suppose f_j (resp. g_j) is the face on the left (resp. right) of e_j . Suppose u_i^∂ is incident to a black internal vertex. By construction of $G^\square$, u_i^∂ is also incident to a black internal vertex. Moreover, we have

$$f_1 = f_{i,\bar{i}+1}^\partial, g_1 = f_{\bar{i}-1,\bar{i}}^\partial, f_r = f_{i-1,i}^\partial, g_r = f_{i,i+1}^\partial, g_1 = g_2, f_2 = f_3, \dots, g_{r-1} = g_r.$$

The alternating product simplifies after a telescopic cancellation to

$$m_i([\text{wt}]) = \sqrt{A_{f_1} A_{g_1}} \frac{1}{A_{f_2} A_{g_2}} \cdots A_{f_{r-1}} A_{g_{r-1}} \frac{1}{\sqrt{A_{f_r} A_{g_r}}} = \sqrt{\frac{A_{f_1} A_{g_r}}{A_{g_1} A_{f_r}}} = \sqrt{\frac{A_{f_{i,i+1}^\partial} A_{f_{\bar{i},\bar{i}+1}^\partial}}{A_{f_{i-1,i}^\partial} A_{f_{i-1,\bar{i}}^\partial}}}.$$

The case when u_i^∂ is incident to a white internal vertex follows by a similar argument, or more easily by color-change equals spider moves at square faces and that $m_i([\text{wt}])$ is invariant under moves. $\square$

Definition 5.9 (Inverse Ising twist map). We define the *inverse Ising twist map* $\hat{\tau}_{\mathcal{M}_\pi}^{-1} : \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0} \rightarrow \Pi_{\mathcal{M}_\pi}^{>0} / T_\pi^{>0}$ as follows:

$$\hat{\tau}_{\mathcal{M}_\pi}^{-1}(V) := \boldsymbol{\lambda} \cdot \bar{\tau}_{\mathcal{M}_\pi}(V),$$

where $\boldsymbol{\lambda} \in \mathbb{R}_{>0}^{2n}$ is any collection of numbers satisfying

$$(15) \quad \lambda_i \lambda_{\bar{i}} = \frac{\Delta_{I_{i+1}}(V)}{\Delta_{I_i}(V)} \quad \text{for all } i \in [2n],$$

Different choices of $\boldsymbol{\lambda}$ are related by the action of $T_\pi^{>0}$ and therefore the definition is independent of the choice.

Remark 5.10. Let G be a reduced graph with $\pi_G = \pi$, let $G^\square$ be the corresponding plabic graph and let $[\text{wt}] = \text{Meas}_{G^\square}^{-1}(V)$. Then, (15) is also equal to $1/m_i([\text{wt}])$, which can be proved by a similar computation as in the proof of Lemma 5.8 applied to p_Γ instead of $\hat{p}_\Gamma$ and using Lemma 2.5(2).

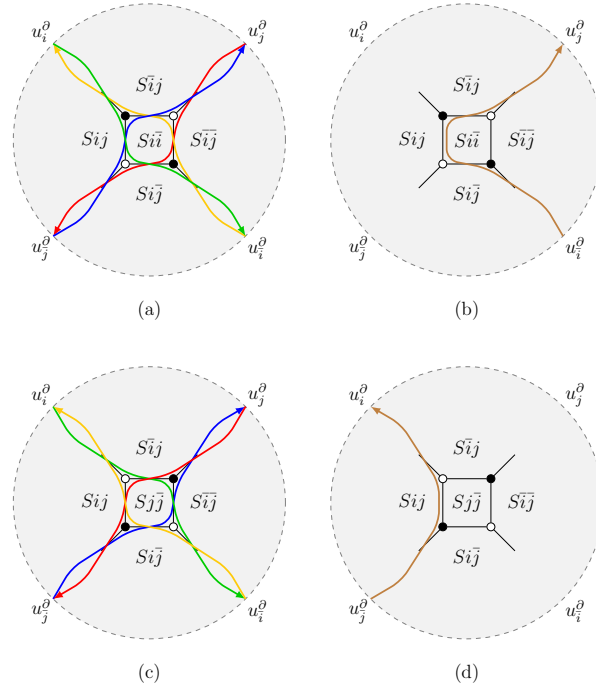


FIGURE 16. Configuration of strands around a square in $G^\square$ and $\tilde{G}^\square$ in the proof of Lemma 5.11.

Lemma 5.11. *We have $\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}) \subset \Lambda_\pi^{>0}/T_\pi^{>0}$.*

Proof. Let $V \in \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}$ and let $U = \hat{\tau}_{\mathcal{M}_\pi}^{-1}(V)$. We need to check (11) for all $I \in \mathcal{S}_\pi^\square$. In order to do this, we use the inverse

$$\tilde{\mathbb{M}}_{G^\square} : \mathcal{X}_{G^\square} \rightarrow \mathcal{A}_{G^\square}$$

of the map $p_{G^\square}$ from [29]. Here we take $G^\square$ to be a plabic graph such that $I \in \mathcal{S}_G^\square$. The map $\tilde{\mathbb{M}}_{G^\square}$ is defined as follows. Let α_i denote the strand with target u_i^∂ .

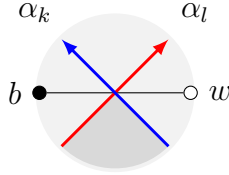


FIGURE 17. Local configuration of strands near an edge $e = bw$.

We first associate to each face f an almost perfect matching M_f with $\partial M_f = \mathbf{T}(f)$ as follows. For an edge $e = bw$ containing strands α_k, α_l as in Figure 17, we let $e \in M_f$ if and only if f is in the *upstream wedge*, i.e., $k \in \mathbf{T}(f)$ and $l \notin \mathbf{T}(f)$ (the shaded region). We then define

$$\tilde{\mathbb{M}}_{G^\square}([\text{wt}]) := \left(\frac{1}{\text{wt}(M_f)} \right)_{f \in F(G^\square)}.$$

The target face labels of faces in $G^\square$ near the square face are as shown in Figure 16(a). We denote by f_{kl} the face with target face label $Sk l$. Let $[\text{wt}] = \text{Meas}_{G^\square}^{-1}(V)$ and $A = \tilde{\mathbb{M}}_{G^\square}([\text{wt}])$. Viewing each edge $e = bw$ of a matching as a 1-chain in $G^\square$ oriented from b to w , we have that $M_{f_{i\bar{i}}} - M_{f_{ij}}$ is a 1-chain with boundary $u_j^\partial - u_i^\partial$. Moreover, since $M_{f_{i\bar{i}}}$ and $M_{f_{ij}}$ only differ by edges in $\alpha_{\bar{i}}$ and α_j , $M_{f_{i\bar{i}}} - M_{f_{ij}}$ is supported on $\alpha_{\bar{i}} \cup \alpha_j$, so it must be the path from u_i^∂ to u_j^∂ in $\alpha_{\bar{i}} \cup \alpha_j$ (Figure 16(b)) and $\frac{A_{f_{ij}}}{A_{f_{i\bar{i}}}}$ is the alternating product of edge weights along this path. Applying the same argument to $\bar{G}^\square$ (Figure 16(c)), we get that $M_{f_{ij}} - M_{f_{j\bar{j}}}$ is the path from u_j^∂ to u_i^∂ in $\alpha_i \cup \alpha_{\bar{j}}$ (Figure 16(d)) and $\frac{A_{f_{j\bar{j}}}}{A_{f_{ij}}}$ is the alternating product of edge weights along this path. Since $[\text{wt}] \in \mathcal{I}_G^{>0}$, the alternating product of edge weights along any segment of α_i is equal to the alternating product of edge weights along the corresponding segment of $\alpha_{\bar{i}}$ and also the corresponding segment of α_i in $\bar{G}^\square$. This implies that

$$\frac{A_{f_{j\bar{j}}}}{A_{f_{i\bar{i}}}} = \frac{A_{f_{ij}}}{A_{f_{i\bar{i}}}} \frac{A_{f_{j\bar{j}}}}{A_{f_{ij}}} = \frac{m_j([\text{wt}])}{m_i([\text{wt}])}.$$

Acting using λ as in (15), we get

$$\frac{\Delta_{Sj\bar{j}}(U)}{\Delta_{Si\bar{i}}(U)} = \frac{\lambda \cdot A_{f_{j\bar{j}}}}{\lambda \cdot A_{f_{i\bar{i}}}} = \frac{\lambda_j \lambda_{\bar{j}}}{\lambda_i \lambda_{\bar{i}}} \frac{A_{f_{j\bar{j}}}}{A_{f_{i\bar{i}}}} = 1. \quad \square$$

By Lemma 5.11, we have a well-defined map

$$\hat{\tau}_{\mathcal{M}_\pi}^{-1} : \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0} \rightarrow \Lambda_\pi^{>0}/T_\pi^{>0}.$$

5.4. Proof of the main theorem. We are now ready to prove our main result.

Theorem 5.12. *The map $\hat{\tau}_{\mathcal{M}_\pi} : \Lambda_\pi^{>0}/T_\pi^{>0} \rightarrow \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0}$ is a bijection whose inverse is $\hat{\tau}_{\mathcal{M}_\pi}^{-1}$. They fit into the following commutative diagram:*

$$\begin{array}{ccc}
 \mathcal{C}_G^{>0}/T_\pi^{>0} & \xrightarrow{q_G} & \mathcal{I}_G^{>0} \\
 \uparrow \mathbf{H}_G & \swarrow \hat{\tau}_{\mathcal{M}_\pi}^{-1} & \downarrow \phi_n \circ \text{Corr}_G \\
 \Lambda_\pi^{>0}/T_\pi^{>0} & & \text{OGr}_{n,2n}^{\geq 0} \cap \Pi_{\mathcal{M}_\pi}^{>0} \\
 & \searrow \hat{\tau}_{\mathcal{M}_\pi} &
 \end{array}$$

In particular, the composition

$$q_G \circ \mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1} \circ \phi_n = \text{Corr}_G^{-1}$$

is the inverse of the boundary correlation map Corr_G .

Proof. The maps $\hat{\tau}_{\mathcal{M}_\pi}$ and $\hat{\tau}_{\mathcal{M}_\pi}^{-1}$ are defined by modifying the right and left twists respectively by torus actions. By Lemma 2.5 and Lemma 5.8, these cancel out. $\square$

5.5. Examples. We now compute some examples to illustrate the result of Theorem 5.12. The computations were carried out using the computer algebra system `Macaulay2` [16]. The code used for our computations is available in [11].

Example 5.13 ($n = 2$). Consider the $n = 2$ Ising model in Figure 11(a). We compute the composition $q_G \circ \mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1} \circ \phi_n$ and verify that it is the inverse of Corr_G . Let

$$M = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \in \text{Mat}_2^{\text{sym}}(\mathbb{R}, 1).$$

Applying ϕ_2 , we obtain

$$\tilde{M} = \begin{bmatrix} 1 & 1 & r & -r \\ -r & r & 1 & 1 \end{bmatrix}$$

with Plücker coordinates

$$\Delta_{12} = \Delta_{34} = 2r, \quad \Delta_{14} = \Delta_{23} = 1 - r^2, \quad \Delta_{13} = \Delta_{24} = 1 + r^2.$$

The reverse Grassmann necklace is

$$\mathbf{I}_\Gamma^{\text{rev}} = (14, 12, 23, 34).$$

Using the definition of the left twist, we get

$$\tau_{\mathcal{M}_\pi}(\tilde{M}) = \begin{bmatrix} \frac{1}{1-r^2} & \frac{1}{2} & -\frac{r}{1-r^2} & -\frac{1}{2r} \\ \frac{r}{1-r^2} & \frac{1}{2r} & \frac{1}{1-r^2} & \frac{1}{2} \end{bmatrix}.$$

We now apply the torus factor to obtain $\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M})$. The Grassmann necklace is

$$\mathbf{I}_\Gamma = (12, 23, 34, 14),$$

and the matching is

$$\bar{1} = 3, \quad \bar{2} = 4.$$

Thus, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ must satisfy

$$\lambda_1 \lambda_3 = \frac{\Delta_{23}}{\Delta_{12}} = \frac{1 - r^2}{2r}, \quad \lambda_2 \lambda_4 = \frac{\Delta_{34}}{\Delta_{23}} = \frac{2r}{1 - r^2}.$$

Taking

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \frac{1-r^2}{2r} \quad \text{and} \quad \lambda_4 = \frac{2r}{1-r^2},$$

we get

$$\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M}) = \boldsymbol{\lambda} \cdot \tilde{\tau}_{\mathcal{M}_\pi}(\tilde{M}) = \begin{bmatrix} \frac{1}{1-r^2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{1-r^2} \\ \frac{r}{1-r^2} & \frac{1}{2r} & \frac{1}{2r} & \frac{r}{1-r^2} \end{bmatrix}.$$

Recall that this is an element in $\Lambda_\pi^{>0}/T_\pi^{>0}$. Up to the action of $T_\pi^{>0}$, the Plücker coordinates of this point are given by:

$$\Delta_{12} = \Delta_{23} = \Delta_{34} = \frac{1}{2r}, \quad \Delta_{13} = \Delta_{24} = \frac{1+r^2}{2r(1-r^2)}, \quad \Delta_{14} = \frac{2r}{(1-r^2)^2}.$$

So after applying $\mathbf{H}_G$ (see Figure 15), we compute $C \in \mathcal{C}_G^{>0}/T_\pi^{>0}$ and get:

$$C_{b_1^\partial} = C_f = C_g = \frac{1}{2r} \quad \text{and} \quad C_{b_2^\partial} = \frac{2r}{(1-r^2)^2}.$$

Finally, applying the Ising cluster ensemble map q_G , we get

$$c_e = \sqrt{\frac{C_{b_1^\partial} C_{b_2^\partial}}{C_{b_1^\partial} C_{b_2^\partial} + C_f C_g}} = \frac{2r}{1+r^2}, \quad s_e = \sqrt{\frac{C_f C_g}{C_{b_1^\partial} C_{b_2^\partial} + C_f C_g}} = \frac{1-r^2}{1+r^2}.$$

Thus, the composition $q_G \circ \mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1} \circ \phi_2(M)$ is the map

$$\begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \mapsto (c_e, s_e) = \left(\frac{2r}{1+r^2}, \frac{1-r^2}{1+r^2} \right),$$

which corresponds to the Ising model with coupling

$$J = \frac{1}{2} \operatorname{arctanh}(c_e) = \frac{1}{2} \operatorname{arctanh}\left(\frac{2r}{1+r^2}\right) = \operatorname{arctanh}(r).$$

By Example 3.4, Corr_G sends the Ising model with coupling constant J to

$$\begin{bmatrix} 1 & \tanh J \\ \tanh J & 1 \end{bmatrix}.$$

Hence, $q_G \circ \mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1} \circ \phi_2$ is the inverse of Corr_G in this example.

Example 5.14 ($n = 4$). Consider the $n = 4$ Ising model in Figure 18. The Grassmann necklace is

$$\mathbf{I}_\Gamma = (1234, 2346, 3467, 4567, 5678, 3678, 1378, 1238),$$

and the reverse necklace is

$$\mathbf{I}_\Gamma^{\text{rev}} = (5678, 1578, 1258, 1238, 1234, 1245, 2456, 4567).$$

The boundary measurement map computed from the edge weights is

$$\text{Meas} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{cf}{de} & -\frac{dh+f}{deg} & -\frac{fh+d}{deg} \\ 0 & 1 & 0 & 0 & 0 & \frac{c}{de} & \frac{dfh+1}{deg} & \frac{df+h}{deg} \\ 0 & 0 & 1 & 0 & -\frac{b}{a} & -\frac{1}{ad} & -\frac{c}{adg} & -\frac{ch}{adg} \\ 0 & 0 & 0 & 1 & \frac{1}{a} & \frac{b}{ad} & \frac{bc}{adg} & \frac{bch}{adg} \end{bmatrix}.$$

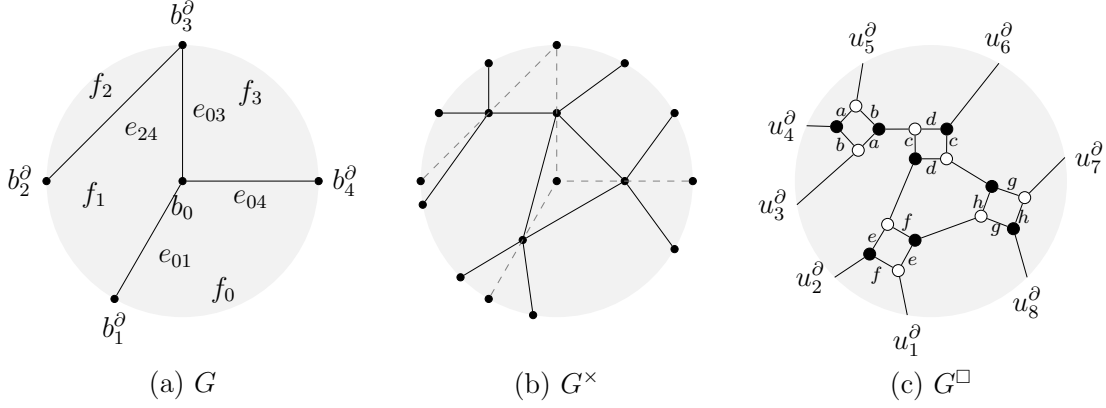


FIGURE 18. An Ising model with $n = 4$ boundary vertices (a), its medial graph (b) and the corresponding dimer model (c). Here the parameters a, b, c, d, e, f, g, h are given in terms of the coupling constants as follows:
 $a = \tanh(2J_{23}), \quad c = \tanh(2J_{03}), \quad e = \tanh(2J_{01}), \quad g = \tanh(2J_{04})$
 $b = \operatorname{sech}(2J_{23}), \quad d = \operatorname{sech}(2J_{03}), \quad f = \operatorname{sech}(2J_{01}), \quad h = \operatorname{sech}(2J_{04}).$

We compute the correlation matrix using [12, Lemma 3.2] and we get

$$M := \phi_4^{-1}(\text{Meas}) = \begin{bmatrix} 1 & \frac{a(1-d)(1-f)}{ce(b+1)} & \frac{c(1-f)}{e(d+1)} & \frac{e(1-h)}{g(f+1)} \\ \frac{a(1-d)(1-f)}{ce(b+1)} & 1 & \frac{1-b}{a} & \frac{a(1-d)(1-h)}{gc(b+1)} \\ \frac{c(1-f)}{e(d+1)} & \frac{1-b}{a} & 1 & \frac{c(1-h)}{g(d+1)} \\ \frac{e(1-h)}{g(f+1)} & \frac{a(1-d)(1-h)}{gc(b+1)} & \frac{c(1-h)}{g(d+1)} & 1 \end{bmatrix}.$$

So the matrix $\tilde{M}$ is given by

$$\tilde{M} = \begin{bmatrix} 1 & 1 & \frac{a(1-d)(1-f)}{ce(b+1)} & -\frac{a(1-d)(1-f)}{ce(b+1)} & -\frac{c(1-f)}{e(d+1)} & \frac{c(1-f)}{e(d+1)} & \frac{e(1-h)}{g(f+1)} & -\frac{e(1-h)}{g(f+1)} \\ -\frac{a(1-d)(1-f)}{ce(b+1)} & -\frac{a(1-d)(1-f)}{ce(b+1)} & 1 & 1 & \frac{1-b}{a} & -\frac{1-b}{a} & -\frac{a(1-d)(1-h)}{gc(b+1)} & \frac{a(1-d)(1-h)}{gc(b+1)} \\ \frac{c(1-f)}{e(d+1)} & -\frac{c(1-f)}{e(d+1)} & -\frac{1-b}{a} & \frac{1-b}{a} & 1 & 1 & \frac{c(1-h)}{g(d+1)} & -\frac{c(1-h)}{g(d+1)} \\ -\frac{e(1-h)}{g(f+1)} & \frac{e(1-h)}{g(f+1)} & \frac{a(1-d)(1-h)}{gc(b+1)} & -\frac{a(1-d)(1-h)}{gc(b+1)} & -\frac{c(1-h)}{g(d+1)} & \frac{c(1-h)}{g(d+1)} & 1 & 1 \end{bmatrix}.$$

Note that the matrices Meas and $\tilde{M}$ define the same element in $\text{OGr}(4, 8) \cap \Pi_{\mathcal{M}_\pi}^{>0}$. Applying the left twist to this matrix we get the following

$$\tilde{\tau}_{\mathcal{M}}(\tilde{M}) = \begin{bmatrix} \frac{f+1}{2f} & \frac{1}{2} & 0 & 0 & 0 & -\frac{ed}{2fc} & -\frac{g(f+1)}{2he} & -\frac{(f+1)(d+h)}{2ged} \\ \frac{ea}{2fc(1-b)} & \frac{(b+1)(df+h)}{2heca} & \frac{b+1}{2b} & \frac{1}{2} & -\frac{a}{2b} & 0 & 0 & -\frac{hca}{2gd(1-b)} \\ -\frac{ed}{2fc} & -\frac{dh+f}{2hec} & \frac{a}{2b} & \frac{b+d}{2da} & \frac{bd+1}{2db} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{gf}{2he} & 0 & \frac{gcb}{2da(1-h)} & \frac{gc}{2db(1-h)} & \frac{g(d+f)}{2fc(1-h)} & \frac{h+1}{2h} & \frac{1}{2} \end{bmatrix}.$$

We then deduce that λ must satisfy the following equations

$$\begin{aligned} \lambda_1 \lambda_6 &= \frac{\Delta_{2346}}{\Delta_{1234}} = \frac{\Delta_{1378}}{\Delta_{3678}} = \frac{cf}{de}, & \lambda_2 \lambda_7 &= \frac{\Delta_{3467}}{\Delta_{2346}} = \frac{\Delta_{1238}}{\Delta_{1378}} = \frac{eh}{fg}, \\ \lambda_3 \lambda_5 &= \frac{\Delta_{4567}}{\Delta_{3467}} = \frac{\Delta_{3678}}{\Delta_{5678}} = \frac{b}{a}, & \lambda_4 \lambda_8 &= \frac{\Delta_{5678}}{\Delta_{4567}} = \frac{\Delta_{2346}}{\Delta_{1238}} = \frac{adg}{bch}, \end{aligned}$$

We can then choose

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1 \quad \text{and} \quad \lambda_5 = \frac{b}{a}, \quad \lambda_6 = \frac{cf}{de}, \quad \lambda_7 = \frac{eh}{fg}, \quad \lambda_8 = \frac{adg}{bch}.$$

Computing $\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M}) = \lambda \cdot \tilde{\tau}_{\mathcal{M}_\pi}(\tilde{M})$ yields the following:

$$\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M}) = \begin{bmatrix} \frac{f+1}{2f} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{f+1}{2f} & -\frac{ea(d+h)}{2hcb(1-f)} \\ \frac{e(b+1)}{2fca} & \frac{a(df+h)}{2hec(1-b)} & \frac{b+1}{2b} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{b+1}{2b} \\ -\frac{ed}{2fc} & -\frac{dh+f}{2hec} & \frac{a}{2b} & \frac{b+d}{2da} & \frac{bd+1}{2da} & \frac{fc}{2ed} & 0 & 0 \\ 0 & -\frac{gf}{2he} & 0 & \frac{(h+1)cb}{2gda} & \frac{(h+1)c}{2gda} & \frac{(h+1)(d+f)}{2ged} & \frac{ge}{2f(1-h)} & \frac{gda}{2hcb} \end{bmatrix}.$$

We recall that this is an element of $\Lambda_\pi^{>0}/T_\pi^{>0}$. We now compute the C -variables which are indexed by the vertices and faces of G in Figure 18(a). They correspond to the Plücker coordinates of $\hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M})$ indexed by the labels of the non-square faces of the plabic graph in Figure 19. We obtain

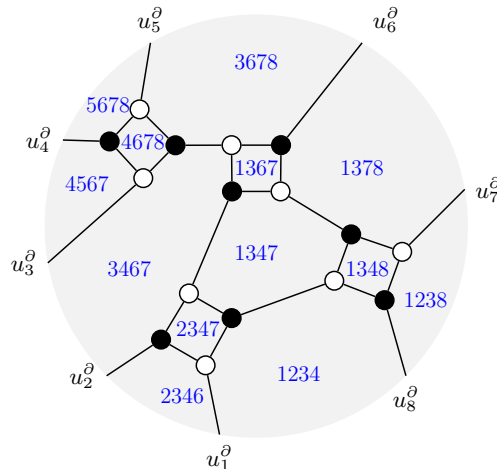
$$\begin{aligned} C_{b_0} &= \Delta_{1347} = \frac{(h+1)(f+1)e(d+1)(b+1)}{8gf^2da}, & C_{b_1^\partial} &= \Delta_{2346} = \frac{g(f+1)(d+1)(b+1)}{8(1-h)eda}, \\ C_{b_2^\partial} &= \Delta_{1238} = \frac{(h+1)g(f+1)d(b+1)a}{8h^2e(1-d)b^2}, & C_{b_3^\partial} &= \Delta_{3678} = \frac{(h+1)(f+1)(d+1)(b+1)a}{8gedb^2}, \\ C_{b_4^\partial} &= \Delta_{4567} = \frac{ge(d+1)(b+1)}{8(1-h)(1-f)da}, & C_{f_0} &= \Delta_{1234} = \frac{g(f+1)(d+1)(b+1)}{8(1-h)eda}, \\ C_{f_1} &= \Delta_{1378} = \frac{(h+1)(f+1)ed(b+1)a}{8gf^2(1-d)b^2}, & C_{f_2} &= \Delta_{3467} = \frac{ge(d+1)(b+1)}{8(1-h)(1-f)da}, \\ C_{f_3} &= \Delta_{5678} = \frac{g(f+1)(d+1)(b+1)}{8(1-h)eda}. \end{aligned}$$

We have now computed the map $\mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M})$. To recover the edge weights in Figure 18(c) from the C -variables, we compute $q_G \circ \mathbf{H}_G \circ \hat{\tau}_{\mathcal{M}_\pi}^{-1}(\tilde{M})$ to get:

$$\begin{aligned} a &= \sqrt{\frac{C_{b_2^\partial} C_{b_3^\partial}}{C_{b_2^\partial} C_{b_3^\partial} + C_{f_1} C_{f_2}}}, & c &= \sqrt{\frac{C_{b_0} C_{b_3^\partial}}{C_{b_0} C_{b_3^\partial} + C_{f_1} C_{f_3}}}, & e &= \sqrt{\frac{C_{b_0} C_{b_1^\partial}}{C_{b_0} C_{b_1^\partial} + C_{f_0} C_{f_1}}}, & g &= \sqrt{\frac{C_{b_0} C_{b_4^\partial}}{C_{b_0} C_{b_4^\partial} + C_{f_0} C_{f_3}}}, \\ b &= \sqrt{\frac{C_{f_1} C_{f_2}}{C_{b_2^\partial} C_{b_3^\partial} + C_{f_1} C_{f_2}}}, & d &= \sqrt{\frac{C_{f_1} C_{f_3}}{C_{b_0} C_{b_3^\partial} + C_{f_1} C_{f_3}}}, & f &= \sqrt{\frac{C_{f_0} C_{f_2}}{C_{b_0} C_{b_1^\partial} + C_{f_0} C_{f_1}}}, & h &= \sqrt{\frac{C_{f_0} C_{f_3}}{C_{b_0} C_{b_4^\partial} + C_{f_0} C_{f_3}}}. \end{aligned}$$

The coupling constants J_{01}, J_{02}, J_{03} and J_{34} are then obtained as follows:

$$J_{01} = \frac{1}{2} \operatorname{arctanh}(e), \quad J_{04} = \frac{1}{2} \operatorname{arctanh}(g), \quad J_{03} = \frac{1}{2} \operatorname{arctanh}(c), \quad \text{and} \quad J_{23} = \frac{1}{2} \operatorname{arctanh}(a).$$

FIGURE 19. The target labels $\mathbf{T}(f)$ for $f \in F(G^\square)$.

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YASSINE ELMAAZOUZ (CALTECH)
Email address: maazouz@caltech.edu

TERRENCE GEORGE (TIFR-CAM)
Email address: terrence@tifrbng.res.in