

Dimers and Beauville integrable systems

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Abstract

Associated to a convex integral polygon N in the plane are two integrable systems: the cluster integrable system of Goncharov and Kenyon constructed from the planar dimer model, and the Beauville integrable system, associated with the toric surface of N . There is a birational map, called the spectral transform, between the phase spaces of the two integrable systems. When N is the triangle $\text{Conv}\{(0, 0), (d, 0), (0, d)\}$, we show that the spectral transform is a birational isomorphism of integrable systems.

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1 Introduction

From the convex integral polygon $N = \text{Conv}\{(0, 0), (d, 0), (0, d)\}$, Goncharov and Kenyon [GK13] construct an algebraic integrable system from the dimer model in the torus as follows: Let Γ be the $d \times d$ fundamental domain for the hexagonal lattice shown in Figure 1. Let \mathcal{X} denote the space of edge-weights on Γ modulo a natural gauge equivalence. The space \mathcal{X} is a cluster \mathcal{X} -variety as defined in [FG09], and therefore has a canonical cluster Poisson structure [GSV03]. Goncharov and Kenyon [GK13] show that symplectic leaves of \mathcal{X} with its cluster Poisson structure are algebraic completely integrable systems, where the Hamiltonians are certain partition functions for dimer covers (i.e. perfect matchings) on Γ .

On the other hand, associated to the convex integral polygon N is the projective toric surface $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$, with the Poisson structure $\theta = z \frac{\partial}{\partial z} \wedge w \frac{\partial}{\partial w}$, where z, w are coordinates on the dense torus $(\mathbb{C}^\times)^2 \subset \mathbb{P}^2$. Bottacin [Bot98], generalizing earlier work of Beauville [Bea91], showed that θ induces a symplectic structure on a dense open subset of $\text{Sym}^g \mathbb{P}^2$ (the g -fold symmetric product, i.e. $(\mathbb{P}^2)^g/S_g$), where $g = \binom{d-1}{2}$ is the genus of a degree d plane curve. Moreover, $\text{Sym}^g \mathbb{P}^2$ with this symplectic structure is an algebraic completely integrable system ([Bea91] and [Bot98, Section 4]), called a Beauville integrable system.

In their study of the moduli space of simple Harnack curves in \mathbb{P}^2 , Kenyon and Okounkov [KO06] constructed a rational map called the spectral transform, which maps symplectic leaves of \mathcal{X} to a finite cover of $\text{Sym}^g \mathbb{P}^2$. Fock [Foc15] proved that the spectral transform is birational.

The main result of this paper is a proof of the following observation of Goncharov and Kenyon (cf. [GK13, Theorem 1.4]):

Theorem 1.1. *The spectral transform is a birational isomorphism of integrable systems.*

It is immediate from the construction of the spectral transform that it maps Hamiltonians of the dimer integrable system to Hamiltonians of the finite cover of the Beauville integrable system. Therefore, the only thing to be proved is that the spectral transform is Poisson.

We now briefly describe the construction of the spectral transform and discuss the main idea. Associated to the graph Γ with edge-weights is a periodic finite-difference operator $K(z, w)$ called the Kasteleyn matrix, whose entries are Laurent polynomials in z, w . The cokernel \mathcal{L} of the Kasteleyn matrix is the push-forward of a line bundle on the degree d curve $\{\det K(z, w) = 0\}$, which is called the spectral curve. The point in $\text{Sym}^g \mathbb{P}^2$ is the divisor of zeroes of a section of this line bundle. A key observation in this paper is that the Kasteleyn matrix gives a resolution of \mathcal{L} by locally free sheaves, which lets us bring in the tools of homological algebra. Since we are comparing Poisson structures, we need to understand the map on tangent spaces given by the differential of the spectral transform. The tangent space to the Beauville system is isomorphic to $\text{Ext}^1(\mathcal{L}, \mathcal{L})$, and we can use the locally free resolution of \mathcal{L} given by the Kasteleyn matrix to get a description of this space that has a natural combinatorial interpretation. Indeed with this description, the differential of the spectral transform is just the identity map (cf. Section 8).

[GK13, Theorem 1.4] is stated more generally for any convex integral polygon N , in which case \mathbb{P}^2 must be replaced by the projective toric surface associated to N . We decided to restrict ourselves to the simplest example of the hexagonal lattice in this paper to focus on the algebro-geometric constructions without getting distracted by combinatorial complications, but our constructions work generally.

Theorem 1.1 was proved for the pentagram map, which is known to be a special case of the Goncharov-Kenyon integrable system [FM16], by Soloviev [Sol13] using analytic techniques. A similar result for the Hitchin integrable system was proved by Biswas, Bottacin and Gómez in [BBG2112] using algebraic techniques that are more similar to the ones we use in this paper.

Conventions

For any complex, we will denote the degree 0 term by underlining it. For a double complex, we underline the term in degree $(0, 0)$. Moreover, for a coherent sheaf \mathcal{F} on a scheme or a stack X , we denote by $H^i(X, \mathcal{F})$ the i th global sections cohomology. For a complex F^\bullet , we denote by $\mathcal{H}^i(F^\bullet)$ its cohomology in degree i . For a double complex $F^{\bullet, \bullet}$, we denote by $\mathbb{H}^i(F^{\bullet, \bullet})$ its hypercohomology in degree i , i.e. \mathcal{H}^i of its total complex.

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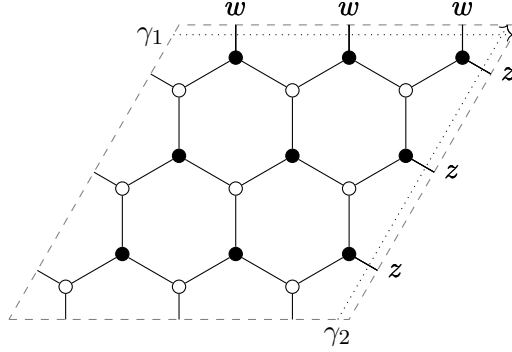


Figure 1: The 3×3 -fundamental domain of the hexagonal lattice showing the cycles γ_1, γ_2 .

2 The dimer integrable system

In this section, we introduce the integrable system associated with the dimer model, following [GK13]. In this paper, we will focus on the simplest setting where Γ is a $d \times d$ -fundamental domain of the hexagonal lattice, where the constructions simplify. We do not require the general definitions, but they can be found in [GK13].

Let $\Gamma = (B \sqcup W, E)$ be the $d \times d$ -fundamental domain of the hexagonal lattice in the torus \mathbb{T} . Here B, W and E denote the set of black vertices, white vertices and edges of Γ respectively.

2.1 Discrete local systems

A (discrete) \mathbb{C}^\times -local system x on Γ is a copy \mathbb{C}_v of \mathbb{C} associated to each vertex $v \in B \sqcup W$, and for each edge $e = bw$, an isomorphism $x(e) : \mathbb{C}_b \rightarrow \mathbb{C}_w$. Since $\text{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$, we will consider $x(e)$ to be an element of \mathbb{C}^\times . Two \mathbb{C}^\times -local systems x_1, x_2 are *isomorphic* if there exists isomorphisms $g(v) : \mathbb{C}_v \rightarrow \mathbb{C}_v$ such that for every edge $e = bw$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}_b & \xrightarrow{x_1(e)} & \mathbb{C}_w \\ \downarrow g(b) & & \downarrow g(w) \\ \mathbb{C}_b & \xrightarrow{x_2(e)} & \mathbb{C}_w \end{array}$$

Let $\text{Loc}(\Gamma, \mathbb{C}^\times)$ denote the space of \mathbb{C}^\times -local systems on Γ modulo isomorphisms. Equivalently, a \mathbb{C}^\times -local system is the same thing as a cellular cohomology cocycle in $Z^1(\Gamma, \mathbb{C}^\times)$, and two \mathbb{C}^\times -local systems are isomorphic if they differ by a coboundary, so $\text{Loc}(\Gamma, \mathbb{C}^\times) \cong H^1(\Gamma, \mathbb{C}^\times)$.

Define $\mathcal{X} := \text{Loc}(\Gamma, \mathbb{C}^\times)$. The space \mathcal{X} is called the *dimer cluster Poisson variety*, and is the phase space of the dimer integrable system. We will describe the Poisson structure on \mathcal{X} in Section 2.4. We denote the isomorphism class of x in \mathcal{X} by $[x]$.

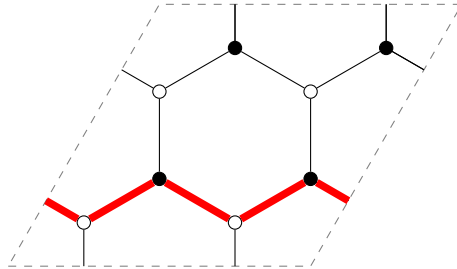


Figure 2: A zig-zag path in Z_2 .

Definition 2.1. A *zig-zag path* in Γ is a closed path that turns maximally right at each black vertex and maximally left at each white vertex (Figure 2). We denote by Z the set of zig-zag paths in Γ . Each zig-zag path $\alpha \in Z$, since it is a closed path in the torus \mathbb{T} , has associated with it a homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z})$. In the basis (γ_1, γ_2) for $H_1(\mathbb{T}, \mathbb{Z})$ shown in Figure 1, it is easy to see that the possible homology classes of zig-zag paths in Γ are either $(-1, 1)$, $(0, -1)$ or $(1, 0)$. We denote by Z_0, Z_1 and Z_2 the sets of zig-zag paths with homology classes $(-1, 1)$, $(0, -1)$ and $(1, 0)$ respectively. Each Z_i consists of d parallel zig-zag paths.

2.2 Tangent space to \mathcal{X}

Let \mathcal{T} denote the group of translations in \mathbb{C} , so we have a canonical isomorphism $\mathcal{T} \cong \mathbb{C}$, $(z \mapsto z + a) \mapsto a$. A \mathbb{C} -local system y on Γ is a line $\mathbb{C}_v \cong \mathbb{C}$ associated to each $v \in B \sqcup W$, and for each edge $e = bw$, a translation $y(e) : \mathbb{C}_b \rightarrow \mathbb{C}_w$. Two discrete line bundles with connection x_1, x_2 are *isomorphic* if there exists translations $g(v) : \mathbb{C}_v \rightarrow \mathbb{C}_v$ such that for every edge $e = bw$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}_b & \xrightarrow{x_1(e)} & \mathbb{C}_w \\ \downarrow g(b) & & \downarrow g(w) \\ \mathbb{C}_b & \xrightarrow{x_2(e)} & \mathbb{C}_w \end{array}$$

Let $\text{Loc}(\Gamma, \mathbb{C})$ denote the space of \mathbb{C} -local systems on Γ modulo isomorphisms. As in the case of \mathbb{C}^\times -local systems, we have $\text{Loc}(\Gamma, \mathbb{C}) \cong H^1(\Gamma, \mathbb{C})$.

Proposition 2.2. *The tangent space $T_{[x]}\mathcal{X}$ is canonically isomorphic to $\text{Loc}(\Gamma, \mathbb{C})$.*

Proof. Let $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ denote the ring of dual numbers. Let x be a \mathbb{C}^\times -local system. A *deformation* of x is a $\mathrm{GL}_1(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ -local system x' on Γ such that $x' \otimes \mathbb{C} = x$, i.e. we recover x from x' by letting $\varepsilon = 0$. This implies that $x' = x + \varepsilon \dot{x}$, where $\dot{x}(e) \in \mathbb{C}$ for all $e \in E$. More explicitly, we have a copy of $\mathbb{C}_v \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$ associated to each $v \in B \sqcup W$, and for each edge $e = bw$, an isomorphism $x(e) + \varepsilon \dot{x}(e) : \mathbb{C}_b \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \rightarrow \mathbb{C}_w \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Then $\frac{\dot{x}}{x}$ is a \mathbb{C} -local system.

The deformation x' is *trivial* if there exists an isomorphism $g' : x' \rightarrow x$ such that $g' \otimes \mathbb{C} = 1$. This means $g' = 1 + \varepsilon \dot{g}$ and the diagram

$$\begin{array}{ccc} \mathbb{C}_b \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) & \xrightarrow{x(e) + \varepsilon \dot{x}(e)} & \mathbb{C}_w \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \\ \downarrow 1 + \varepsilon \dot{g}(b) & & \downarrow 1 + \varepsilon \dot{g}(w) \\ \mathbb{C}_b \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) & \xrightarrow{x(e)} & \mathbb{C}_w \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \end{array}$$

commutes, which is equivalent to the equation

$$x(e)\dot{g}(b) + \dot{x}(e) = \dot{x}(e) + \dot{g}(w)x(e),$$

which implies that

$$\frac{\dot{x}(e)}{x(e)} = \dot{g}(b) - \dot{g}(w),$$

which means the \mathbb{C} -local system $\frac{\dot{x}}{x}$ is trivial. \square

2.3 Framed \mathbb{C} -local systems and the cotangent space to \mathcal{X}

Let 0_α denote the trivial \mathbb{C} -local system on the zig-zag path α i.e. the \mathbb{C} -local system $0_\alpha(e) = 0$ for all edges $e \in E \cap \alpha$. A \mathbb{C} -local system y on α is said to be *trivializable* if it is isomorphic to 0_α , i.e. there exists $g : (B \sqcup W) \cap \alpha \rightarrow \mathbb{C}$ such that $g(w) + y(e) - g(b) = 0$ for every edge $e = bw$ in α .

Suppose y is a \mathbb{C} -local system on Γ that is trivializable along every zig-zag path. A *trivialization* of y along a zig-zag path α is a choice of isomorphism

$$y|_\alpha \xrightarrow[\cong]{\phi_\alpha} 0_\alpha$$

of the restriction of y to α with 0_α .

A *(discrete) framed \mathbb{C} -local system* is a pair (y, ϕ) where y is a \mathbb{C} -local system that is trivializable around every zig-zag path, and $\phi = (\phi_\alpha)_{\alpha \in Z}$ is a choice of trivialization around each zig-zag path in Γ . Two framed \mathbb{C} -local systems (y, ϕ) and (y', ϕ') are *isomorphic* if

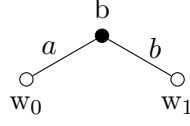


Figure 3: Three consecutive vertices w_0, b, w_1 forming a wedge in α .

there is an isomorphism $y \xrightarrow{g} y'$ such that $\phi'_\alpha \circ g = \phi_\alpha$ for all $\alpha \in Z$. Let $\text{Loc}^{\text{fr}}(\Gamma, \mathbb{C})$ denote the moduli space of framed \mathbb{C} -local systems. There is a canonical map

$$\begin{aligned} \text{pr}_1 : \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) &\rightarrow \text{Loc}(\Gamma, \mathbb{C}) \\ [(y, \phi)] &\mapsto [y] \end{aligned} \quad (1)$$

given by projection onto the first factor.

Concretely, a framed \mathbb{C} -local system consists of $y(e) \in \mathbb{C}$ for each edge $e \in E$ and $\phi_\alpha(v) \in \mathbb{C}$ for each zig-zag path $\alpha \in Z$ and vertex $v \in B \sqcup W$ contained in α , satisfying for each zig-zag path $\alpha \in Z$ and each edge $e = bw$ in α the equation

$$\phi_\alpha(b) = y(e) + \phi_\alpha(w).$$

This equation determines $\phi_\alpha(b)$ in terms of $y(e)$ and $\phi_\alpha(w)$, so we can equivalently define a framed \mathbb{C} -local system as the data of $y(e) \in \mathbb{C}$ for each edge e and $\phi_\alpha(w) \in \mathbb{C}$ for each zig-zag path $\alpha \in Z$ and white vertex $w \in W \cap \alpha$ satisfying

$$\phi_\alpha(w_0) + y(bw_0) = \phi_\alpha(w_1) + y(bw_1), \quad (2)$$

where w_0, b, w_1 is a consecutive sequence of three vertices forming a wedge in α as in Figure 3.

Any edge $e = bw$ is contained in two zig-zag paths. Let $\alpha_r(e)$ (resp. $\alpha_l(e)$) denote the zig-zag path containing e that traverses e from b to w (resp. w to b).

Proposition 2.3. *The cotangent space $T_{[x]}^* \mathcal{X}$ is canonically isomorphic to $\text{Loc}^{\text{fr}}(\Gamma, \mathbb{C})$. The isomorphism is determined by the Poincaré duality pairing*

$$\begin{aligned} \text{Loc}(\Gamma, \mathbb{C}) \otimes \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) &\rightarrow \mathbb{C} \\ [x] \otimes [(y, \phi)] &\mapsto \sum_{e \in E} x(e) (\phi_{\alpha_l(e)}(w) - \phi_{\alpha_r(e)}(w)), \end{aligned}$$

where x and (y, ϕ) are representatives for $[x]$ and $[(y, \phi)]$ respectively.

We will justify the name Poincaré duality for the pairing in the Appendix.

Remark 2.4. We note that the sum

$$\sum_{e \in E} x(e) (\phi_{\alpha_l(e)}(w) - \phi_{\alpha_r(e)}(w)) = \sum_{\alpha \in Z} \sum_{b \in B \cap \alpha} (x(bw_0) \phi_\alpha(w_0) - x(bw_1) \phi_\alpha(w_1)),$$

where $w_0 b w_1$ is a wedge in α as in Figure 3.

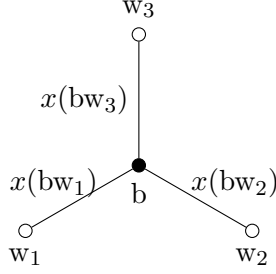


Figure 4: The Kasteleyn matrix.

2.4 Poisson structure on \mathcal{X}

Instead of specifying a Poisson structure as a map $\theta : T^*\mathcal{X} \otimes T^*\mathcal{X} \rightarrow \mathbb{C}$, dually we can give a map $\xi : T^*\mathcal{X} \rightarrow T\mathcal{X}$ so that $\theta(\omega \otimes \eta) = \omega(\xi(\eta))$ and $\xi(\omega) = (\eta \mapsto \theta(\omega \otimes \eta)) \in (T^*\mathcal{X})^* \cong T\mathcal{X}$. The Poisson structure on \mathcal{X} is defined as the composition

$$T_{[x]}^*\mathcal{X} \cong \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \xrightarrow{\text{Pr}_1} \text{Loc}(\Gamma, \mathbb{C}) \cong T_{[x]}\mathcal{X}. \quad (3)$$

We will show in the Appendix that this is the same as the Poisson structure defined in [GK13].

For a loop L in Γ and a \mathbb{C}^\times -local system x , we define the *monodromy* $m_L(x)$ of x around L as the compositions of the isomorphisms $x(e)$ around L . Concretely, if L is the 1-cycle $w_1 \xrightarrow{e_1} b_1 \xrightarrow{e_2} w_2 \xrightarrow{e_3} b_2 \xrightarrow{e_4} \dots \xrightarrow{e_{2n-2}} w_n \xrightarrow{e_{2n-1}} b_n \xrightarrow{e_{2n}} w_1$, then

$$m_L(x) = \prod_{i=1}^n \frac{x(e_{2i})}{x(e_{2i-1})}.$$

Note that m_L is unchanged if we replace x with an isomorphic \mathbb{C}^\times -local system, so $m_L([x])$ is well-defined.

For each zig-zag path α , let $\chi_\alpha \in \mathbb{C}^\times$ be generic nonzero complex numbers satisfying $\prod_{\alpha \in Z} \chi_\alpha = 1$, and let $\chi := (\chi_\alpha)_{\alpha \in Z}$. Let \mathcal{X}_χ denote the subvariety of \mathcal{X} consisting of $[x]$ such that $m_\alpha([x]) = \chi_\alpha$ for all $\alpha \in Z$. The spaces \mathcal{X}_χ are the symplectic leaves of the Poisson structure.

2.5 The Kasteleyn matrix

Let $[x] \in \mathcal{X}$ and let x be a \mathbb{C}^\times -local system in the isomorphism class $[x]$. The *Kasteleyn matrix* is the map of free $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ -modules

$$K(x)(z, w) : \bigoplus_{b \in B} \mathbb{C}[z^{\pm 1}, w^{\pm 1}] \rightarrow \bigoplus_{w \in W} \mathbb{C}[z^{\pm 1}, w^{\pm 1}] \quad (4)$$

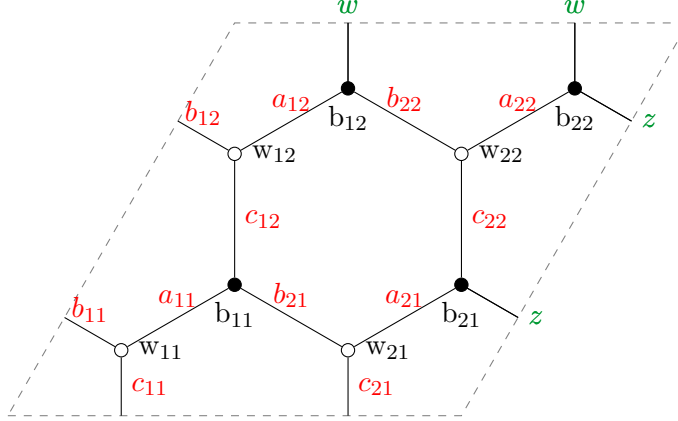


Figure 5: A \mathbb{C}^\times -local system x (red) on a 2×2 -fundamental domain for the honeycomb lattice, along with the z and w factors (green).

defined as

$$K(x)(z, w)_{w,b} := \sum_{e=\{b,w\} \in E} x(e) z^{(e,\gamma_2)} w^{-(e,\gamma_1)},$$

where (\cdot, \cdot) is the intersection index in \mathbb{T} , defined as follows: We orient e from b to w , and define (e, γ_i) to be $+1$ if e crosses γ_i from its left hand side to its right hand side, and -1 otherwise. Concretely, any edge crossing γ_2 gets weighted by an extra factor of z and any edge crossing γ_1 gets weighted by an extra factor of w (see Figure 1). The polynomial $P(z, w) := \det K(x)(z, w)$ is called the *characteristic polynomial* and $C_0 = \{P(z, w) = 0\} \subset (\mathbb{C}^\times)^2$ is called the *open spectral curve*. Consider the embedding of the torus

$$\begin{aligned} (\mathbb{C}^\times)^2 &\hookrightarrow \mathbb{P}^2 \\ (z, w) &\mapsto [1 : z : w]. \end{aligned}$$

The closure $C := \overline{C_0}$ in \mathbb{P}^2 is called the *spectral curve*. Note that while $K(x)$ and $P(z, w)$ depend on the choice of representative x , the curves C_0 and C only depend on $[x]$. For generic $[x] \in \mathcal{X}_\chi$, the polynomial $P(z, w)$ has Newton polygon $N = \text{Conv}\{(0, 0), (d, 0), (0, d)\}$ [KO06, Theorem 1].

Example 2.5. Consider the 2×2 -fundamental domain for the honeycomb lattice shown

in Figure 5. The Kasteleyn matrix is

$$K(z, w) = \begin{bmatrix} b_{11} & b_{12} & b_{21} & b_{22} \\ a_{11} & c_{11}w & b_{11}z & 0 \\ c_{12} & a_{12} & 0 & b_{12}z \\ b_{21} & 0 & a_{21} & c_{21}w \\ 0 & b_{22} & c_{22} & a_{22} \end{bmatrix} \begin{matrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{matrix},$$

and the characteristic polynomial is

$$\begin{aligned} P(z, w) = & a_{11}a_{12}a_{21}a_{22} - (a_{11}a_{12}c_{21}c_{22} + a_{21}a_{22}c_{11}c_{12})w - \\ & (a_{11}a_{21}b_{12}b_{22} + a_{12}a_{22}b_{11}b_{21})z + b_{12}b_{11}b_{21}b_{22}z^2 - \\ & (b_{12}b_{21}c_{11}c_{22} + b_{11}b_{22}c_{12}c_{21})wz + c_{11}c_{12}c_{21}c_{22}w^2. \end{aligned} \quad (5)$$

We normalize $P(z, w)$ so that the coefficient of $1 = z^0w^0$ is 1. Let N° denote the interior of N , and for each lattice point $(i, j) \in N^\circ \cap \mathbb{Z}^2$, let $H_{i,j}$ denote the coefficient of $z^i w^j$ in $P(z, w)$. Let $\mathcal{U} = \mathbb{C}^{N^\circ \cap \mathbb{Z}^2} \cong \mathbb{C}^g$ denote the space of these coefficients, so have a fibration $\mathcal{X}_\chi \rightarrow \mathcal{U}$.

Theorem 2.6 ([GK13]). *For generic χ , the map $\mathcal{X}_\chi \rightarrow \mathcal{U}$ is Lagrangian, so the generic level sets of χ are algebraic integrable systems.*

Remark 2.7. These integrable systems are called dimer integrable systems because the functions $H_{i,j}$ are partition functions for the dimer model in Γ (see for example (5)).

3 Equivariant line bundles on \mathbb{P}^2

The Kasteleyn matrix (4) is defined as a map of trivial locally free sheaves on $(\mathbb{C}^\times)^2$, and we would like to extend it to a map of sheaves on the compactification \mathbb{P}^2 of $(\mathbb{C}^\times)^2$. However, as already observed by Kenyon and Okounkov [KO06, Section 3.1], the construction of a natural extension of the Kasteleyn matrix requires taking d th roots of z, w . In other words, one constructs the extension on a finite branched cover of \mathbb{P}^2 given by the d^2 -fold covering map $\pi : \mathbb{P}^2 \rightarrow \mathbb{P}^2, [x_0 : x_1 : x_2] \mapsto [x_0^d : x_1^d : x_2^d]$ (cf. Proposition 4.1); or equivalently, on a stacky \mathbb{P}^2 . The goal of this section is to collect a few results on G -equivariant line bundles on \mathbb{P}^2 , which is the same thing as line bundles on the stack quotient $[\mathbb{P}^2/G]$, that will be useful later. While some of the proofs require notions of stacks, we have phrased the the statements in terms of G -equivariant line bundles and their G -invariant sections so that they can be understood without any knowledge of stacks.

3.1 Line bundles on $[\mathbb{P}^2/G]$

We begin this subsection by introducing some notations that we will use in the rest of the paper.

Notation 3.1. We denote by $D = V(x_0x_1x_2) \subseteq \mathbb{P}^2$ the complement of the torus in \mathbb{P}^2 . We denote by $\boldsymbol{\mu}_d$ the multiplicative group of d -th roots of unity, and by $G := (\boldsymbol{\mu}_d)^3/\boldsymbol{\mu}_d$ the covering group of π , where we are taking the quotient by the diagonal subgroup of $(\boldsymbol{\mu}_d)^3$.

As G is the covering group of the ramified covering π , we have an action of G on \mathbb{P}^2 given by

$$(\zeta_0, \zeta_1, \zeta_2) \cdot [x_0 : x_1 : x_2] = [\zeta_0x_0 : \zeta_1x_1 : \zeta_2x_2].$$

In this subsection we describe G -equivariant line bundles on \mathbb{P}^2 . For background on equivariant sheaves, see [CG10, Section 5.1] or [MFK94, Chapter 1, §3]. Many of the results here can also be found in [BH09], we report the parts which are relevant for our paper.

Definition 3.2 ([MFK94, Chapter 1, §3]). For every $d \in \mathbb{Z}$, a *linearization* of $\mathcal{O}_{\mathbb{P}^2}(d)$ is a bundle action of G on the total space of $\mathcal{O}_{\mathbb{P}^2}(d)$ (namely, a homomorphism $G \rightarrow \text{Aut}(\mathcal{O}_{\mathbb{P}^2}(d))$, where Aut denotes the group of isomorphisms of line bundles), such that the projection $\mathcal{O}_{\mathbb{P}^2}(d) \rightarrow \mathbb{P}^2$ is equivariant.

Definition 3.3. A *G -equivariant line bundle* $\mathcal{L} \rightarrow \mathbb{P}^2$ is a line bundle on \mathbb{P}^2 together with a choice of a linearization.

It is easy to check that if \mathcal{L}_1 and \mathcal{L}_2 are G -equivariant line bundles, then their tensor product and inverses are also G -equivariant line bundles. This leads to the following definition.

Definition 3.4. We denote by $\text{Pic}^G(\mathbb{P}^2)$ the group of G -equivariant line bundles up to isomorphism.

Remark 3.5. The data of a G -equivariant line bundle is equivalent to the data of a line bundle on the quotient stack $[\mathbb{P}^2/G]$.

We begin by the following:

Notation 3.6. Let H be the subgroup of $\mathbb{C}^\times \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d$, isomorphic to $\boldsymbol{\mu}_d$, given by elements of the form $(\zeta^{-1}, \zeta, \zeta, \zeta)$, and let $\mathcal{G} := (\mathbb{C}^\times \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d)/H$.

It is not hard to check that there is an isomorphism $\mathbb{C}^\times \times G \cong \mathcal{G}$. There is also an injective homomorphism $\Psi : \mathcal{G} \rightarrow (\mathbb{C}^\times)^3$, induced by

$$\mathbb{C}^\times \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d \rightarrow (\mathbb{C}^\times)^3, \quad (\lambda, \zeta_1, \zeta_2, \zeta_3) \mapsto (\lambda\zeta_1, \lambda\zeta_2, \lambda\zeta_3).$$

Notation 3.7. Since $(\mathbb{C}^\times)^3$ acts on $U := \mathbb{C}^3 - \{(0, 0, 0)\}$ by sending $(\lambda, \mu, \nu) * (a, b, c) := (\lambda a, \mu b, \nu c)$, we can restrict this action to \mathcal{G} on U . We denote this action by $*_{\mathcal{G}}$.

The following proposition boils down to the fact that $\mathbb{P}^2 \cong U/\mathbb{C}^\times$, and the definition of \mathcal{G} :

Proposition 3.8. *There is an isomorphism $[\mathbb{P}^2/G] \cong [U/\mathcal{G}]$.*

Proof. Consider K the subgroup of \mathcal{G} given by the elements of the form $(\lambda, 1, 1, 1)$. This is a normal subgroup of G , and we have isomorphisms

$$[U/\mathcal{G}] \cong [(U/K)/(\mathcal{G}/K)] \cong [\mathbb{P}^2/G]$$

where the first isomorphism comes from [Rom22, Proposition 4.5.3], and the last isomorphism follows from the fact that $\mathbb{P}^2 \cong [U/K]$ and $G \cong \mathcal{G}/K$. \square

Corollary 3.9. *There is a one to one correspondence between line bundles on $[\mathbb{P}^2/G]$ and characters of \mathcal{G} .*

Proof. The data of a line bundle on $[\mathbb{P}^2/G]$ is equivalent to the data of a line bundle on $[U/\mathcal{G}]$. Recall that this was just a line bundle $V \rightarrow U$ on U , together with a line bundle action of \mathcal{G} on V , so that $V \rightarrow U$ is equivariant. But as $U = \mathbb{C}^3 - \{(0, 0, 0)\}$, every line bundle on U is trivial, so $V \cong \mathbb{C} \times U$. Therefore the data of a \mathcal{G} -equivariant line bundle on U is the data of a linear action of \mathcal{G} on $V \cong \mathbb{C} \times U$ that makes the second projection $\mathbb{C} \times U \rightarrow U$ a \mathcal{G} -equivariant map. The second projection uniquely determines the action of \mathcal{G} on $\{0\} \times U$, so one has just to specify the action on \mathbb{C} . In other terms, an equivariant line bundle on U is equivalent to the data of a character of \mathcal{G} . \square

Definition 3.10. For each $r = (r_0, r_1, r_2) \in \mathbb{Z}^3$, we denote by $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$ the G -equivariant line bundle on \mathbb{P}^2 given by the following character of \mathcal{G} :

$$(\lambda, \zeta_1, \zeta_2, \zeta_3) * t := \lambda^{r_1+r_2+r_3} \zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3} t.$$

The following is a straightforward computation (see also [BH09, Proposition 3.3]):

Proposition 3.11. [BH09, Proposition 3.3] *The G -equivariant line bundle $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$ is trivial if and only if (r_0, r_1, r_2) is contained in the span of the vectors $(d, 0, -d), (0, d, -d)$.*

In other words,

$$\text{Pic}^G(\mathbb{P}^2) \cong \mathbb{Z}^3 / \langle (d, 0, -d), (0, d, -d) \rangle.$$

Lemma 3.12. *The canonical map $\text{Pic}^G(\mathbb{P}^2) \rightarrow \text{Pic}(\mathbb{P}^2)$ that forgets the G -equivariant structure is given by*

$$\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2) \mapsto \mathcal{O}_{\mathbb{P}^2}(r_0D_0 + r_1D_1 + r_2D_2) \cong \mathcal{O}_{\mathbb{P}^2}(r_0 + r_1 + r_2).$$

Proof. Consider the inclusion $\mathbb{C}^\times \rightarrow \mathcal{G}$, $\lambda \mapsto (\lambda, 1, 1, 1)$. If we restrict the action $*_{\mathcal{G}}$ to \mathbb{C}^\times , we get the usual action with scaling on U that defines \mathbb{P}^2 . But the restriction of the action gives a morphism $[U/\mathbb{C}^\times] \rightarrow [U/\mathcal{G}]$. Now the morphism $\mathbb{P}^2 \rightarrow [\mathbb{P}^2/G]$ comes from

$$\mathbb{P}^2 \cong [U/\mathbb{C}^\times] \rightarrow [U/\mathcal{G}] \cong [\mathbb{P}^2/G].$$

It is now straightforward to compute the pull-back on equivariant Picard groups. Indeed, a line bundle on $[U/\mathcal{G}]$ is a character of \mathcal{G} , a line bundle on $[U/\mathbb{C}^\times]$ is a character of \mathbb{C}^\times , and the restriction $\text{Pic}([U/\mathcal{G}]) \rightarrow \text{Pic}([U/\mathbb{C}^\times])$ is the restriction of characters. It suffices now to observe that the character $(\lambda, \zeta_1, \zeta_2, \zeta_3) \cdot t = \lambda^{r_1+r_2+r_3} \zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3} t$ restricts to $(\lambda, 1, 1, 1) \cdot t = \lambda^{r_1+r_2+r_3} t$. \square

3.2 Sections of $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$

We fix the following notation for the rest of the paper.

We are now interested in global sections of the line bundle $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$. As for toric varieties, these can be described combinatorially, using the following polygon:

Notation 3.13. For every $r \in \mathbb{Z}^3$ we denote by $P(r)$ the two dimensional polygon defined by

$$P(r) := \{-x - y + r_0 \geq 0\} \cap \{x + r_1 \geq 0\} \cap \{y + r_2 \geq 0\}.$$

Proposition 3.14. *Let x_0, x_1, x_2 be the homogeneous coordinates on \mathbb{P}^2 , so that G acts on x_0, x_1, x_2 as $(\zeta_1, \zeta_2, \zeta_3)(x_0, x_1, x_2) = (\zeta_1x_0, \zeta_2x_1, \zeta_3x_2)$. Let then $z := \frac{x_1}{x_0}$ and $w = \frac{x_2}{x_0}$. Then there is a bijection*

$$H^0([\mathbb{P}^2/G], \mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)) = \bigoplus_{(i,j) \in d\mathbb{Z}^2 \cap P(r)} \mathbb{C}\{z^i w^j\}.$$

Proof. We need to find the \mathcal{G} -equivariant sections of the map $\pi_2 : \mathbb{C} \times U \rightarrow U$, $(t, u) \mapsto u$ where \mathcal{G} acts $\mathbb{C} \times U$ as

$$(\lambda, \zeta_1, \zeta_2, \zeta_3) \cdot (t, u_0, u_1, u_2) := (\lambda^{r_1+r_2+r_3} \zeta_1^{r_1} \zeta_2^{r_2} \zeta_3^{r_3} t, \lambda \zeta_0 u_0, \lambda \zeta_1 u_1, \lambda \zeta_2 u_2)$$

and on U via $*_{\mathcal{G}}$. It is then easy to check that a basis for the \mathbb{C} -vector spaces of \mathcal{G} -equivariant sections of π_2 is given by maps $U \rightarrow \mathbb{C} \times U$ of the form

$$(a, b, c) \rightarrow (a^\alpha b^\beta c^\gamma, a, b, c)$$

such that α, β, γ are non-negative, $\alpha + \beta + \gamma = r_0 + r_1 + r_2$, $d|\beta - r_1$, and $d|\gamma - r_2$. If we define $i := \beta - r_1$ and $j := \gamma - r_2$, then an invariant section is a linear combination of sections of the form

$$x_0^{r_0} x_1^{r_1} x_2^{r_2} \left(\frac{x_1}{x_0}\right)^i \left(\frac{x_2}{x_0}\right)^j \quad \text{with} \quad i + r_1 \geq 0, \quad j + r_2 \geq 0, \quad r_0 - i - j \geq 0 \quad \text{and} \quad (i, j) \in d\mathbb{Z}^2.$$

\square

Throughout the rest of the paper we will adopt the following notations:

Notation 3.15. Let $\mathcal{F}_1, \mathcal{F}_2$ be two coherent sheaves on $[\mathbb{P}^2/G]$, or equivalently two G -equivariant sheaves on \mathbb{P}^2 . We will denote by $\text{Ext}^i(\mathcal{F}_1, \mathcal{F}_2)^G$ the vector spaces $\text{Ext}_{[\mathbb{P}^2/G]}^i(\mathcal{F}_1, \mathcal{F}_2)$. Similarly, we will denote by $H^i(\mathbb{P}^2, F_1)^G$ the vector space $H^i([\mathbb{P}^2/G], F_1)$, and by $\mathcal{E}xt^i(\mathcal{F}_1, \mathcal{F}_2)^G$ the sheaves $\mathcal{E}xt_{[\mathbb{P}^2/G]}^i(\mathcal{F}_1, \mathcal{F}_2)$.

3.2.1 Examples

We collect a series of examples that will be used in the rest of the paper.

Example 3.16. Consider the G -equivariant line bundle $\mathcal{O}_{\mathbb{P}^2}^G(D_0)$, so $r = (1, 0, 0)$. The polygon is $P(1, 0, 0)$ the unit triangle: $P(1, 0, 0) = \text{Conv}\{(0, 0), (1, 0), (0, 1)\}$. Therefore we have

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(D_0)) = \mathbb{C}\{1, z, w\} \text{ and } H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_0))^G = \mathbb{C}\{1\}.$$

In homogeneous coordinates, we have $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_0))^G \cong \mathbb{C}\{x_0\}$.

Example 3.17. Consider the G -equivariant line bundle $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$, where $r_0 + r_1 + r_2 = 0$. Then $r_0D_0 + r_1D_1 + r_2D_2 = \text{div } z^{-i}w^{-j}$ is a principal divisor, and we have

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r_0D_0 + r_1D_1 + r_2D_2)) = \mathbb{C}\{z^i w^j\}.$$

Therefore we get

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2))^G = \begin{cases} \mathbb{C}\{z^i w^j\} & \text{if } (i, j) \in d\mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, in homogeneous coordinates, we have $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G)^G \cong \mathbb{C}\{1\}$.

Example 3.18. Suppose $\mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2)$ is a G -equivariant line bundle. There is a canonical isomorphism

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2))^G \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(r_0D_0 + r_1D_1 + r_2D_2 - \text{div } z^i w^j))^G$$

given by $s \mapsto z^i w^j s$.

Let $D = D_0 + D_1 + D_2$. For an equivariant line bundle \mathcal{L} on \mathbb{P}^2 , taking G -invariants in the Serre duality isomorphism

$$H^2(\mathbb{P}^2, \mathcal{L}(-D)) \cong H^0(\mathbb{P}^2, \mathcal{L})^*$$

we get the G -equivariant version of Serre duality

$$H^2(\mathbb{P}^2, \mathcal{L}(-D))^G \cong (H^0(\mathbb{P}^2, \mathcal{L})^G)^*.$$

Example 3.19. We have $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-D)) \cong \frac{1}{x_0 x_1 x_2} \mathbb{C}[\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}]_{-3} = \mathbb{C}\{\frac{1}{x_0 x_1 x_2}\}$. Since $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^G \cong \mathbb{C}$, by G -equivariant Serre duality, we must have $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \cong \mathbb{C}$, so $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \cong \mathbb{C}\{\frac{1}{x_0 x_1 x_2}\}$.

3.3 Properties of the coarse moduli space of $[\mathbb{P}^2/G]$

In this section we report some properties on the push-forward of a line bundle on $[\mathbb{P}^2/G]$ to its coarse moduli space, which we will use later in Section 5.2.

We begin by observing that the coarse moduli space of $[\mathbb{P}^2/G]$ is \mathbb{P}^2 , and the composition $\mathbb{P}^2 \rightarrow [\mathbb{P}^2/G] \xrightarrow{p} \mathbb{P}^2$, where the first map is the universal G -torsor over $[\mathbb{P}^2/G]$ and p is the coarse moduli space, is the morphism π introduced at the beginning of this section.

Notation 3.20. As mentioned above, we denote by $p : [\mathbb{P}^2/G] \rightarrow \mathbb{P}^2$ the coarse moduli space map.

Lemma 3.21. *If \mathcal{L} is a line bundle on $[\mathbb{P}^2/G]$, then $p_*\mathcal{L}$ is a line bundle on \mathbb{P}^2 .*

Before we begin the proof, recall that $p_*\mathcal{L}$ in our case can be defined as the sheaf associated to the pre-sheaf $U \mapsto \mathcal{L}(\pi^{-1}(U))^G$. Namely, the sheaf obtained by taking G -invariant sections.

Proof. Since $p_*\mathcal{L}$ is a torsion-free sheaf, and torsion free sheaves are free over a DVR, $p_*\mathcal{L}$ is a line bundle on codimension one. Namely, there is a line bundle L_U defined on $U := \mathbb{P}^2 \setminus \{x_1, \dots, x_n\}$ that is isomorphic to $p_*\mathcal{L}$ on U . But the restriction map $\text{Pic}(\mathbb{P}^2) \rightarrow \text{Pic}(U)$ is an isomorphism, so we can extend L_U to a line bundle L on \mathbb{P}^2 . Moreover, if we denote by $i : U \rightarrow \mathbb{P}^2$ the inclusion of U , we have $L = i_*L_U$.

Now, let $V := \pi^{-1}(U)$, and consider the following commutative diagram:

$$\begin{array}{ccc} [V/G] & \xrightarrow{j} & [\mathbb{P}^2/G] \\ \downarrow q & & \downarrow p \\ U & \xrightarrow{i} & \mathbb{P}^2 \end{array}$$

Since \mathcal{L} is S_2 , we have $\mathcal{L} = j_*j^*\mathcal{L}$, so $p_*\mathcal{L} = p_*j_*j^*\mathcal{L} = i_*q_*j^*\mathcal{L} = i_*L_U = L$ as desired. \square

In particular, we have a map $\text{Pic}^G(\mathbb{P}^2) \rightarrow \text{Pic}(\mathbb{P}^2)$ given by push forward via p which we now describe in a particular case. The same proof goes through in general, but as we don't need the general case we report our specific situation

Proposition 3.22. *With the notations above, $p_*\mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2) = \mathcal{O}_{\mathbb{P}^2}(-3)$.*

Proof. The idea is the following. If we denote by $\mathcal{L} := \mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2)$, from Lemma 3.21 we know that $p_*\mathcal{L} = L$ is a line bundle, so we can take its inverse, and get $p_*\mathcal{L} \otimes L^\vee = \mathcal{O}_{\mathbb{P}^2}$. But from projection formula, $p_*\mathcal{L} \otimes L^\vee \cong p_*(\mathcal{L} \otimes p^*L^\vee)$. Since we can characterize $\mathcal{O}_{\mathbb{P}^2}$ as the only line bundle on \mathbb{P}^2 with one dimensional global sections, $p_*\mathcal{L}$ is the only line bundle L such that $H^0(p_*(\mathcal{L} \otimes p^*L^\vee)) \cong \mathbb{C}$. We can understand global sections of a line bundle using Proposition 3.14, so we are left with understanding the pull-back map

$\text{Pic}(\mathbb{P}^2) \rightarrow \text{Pic}([\mathbb{P}^2/G])$. The composition $\text{Pic}(\mathbb{P}^2) \rightarrow \text{Pic}([\mathbb{P}^2/G]) \rightarrow \text{Pic}(\mathbb{P}^2)$ agrees with the pull back via π , so it sends $\mathcal{O}_{\mathbb{P}^2}(1) \mapsto \mathcal{O}_{\mathbb{P}^2}(d)$. Moreover, the action of G on $p^*\mathcal{O}_{\mathbb{P}^2}(1)$ is trivial. Therefore $p^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{\mathbb{P}^2}^G(dD_0)$ (which is isomorphic to $\mathcal{O}_{\mathbb{P}^2}^G(dD_1)$ and $\mathcal{O}_{\mathbb{P}^2}^G(dD_2)$). Now, using Proposition 3.14, it is easy to check that

$$H^0(\mathcal{O}_{\mathbb{P}^2}^G(-D_0 + -D_1 + -D_2) \otimes p^*\mathcal{O}_{\mathbb{P}^2}(3)) = H^0(\mathcal{O}_{\mathbb{P}^2}^G((d-1)D_0 + (d-1)D_1 + (d-1)D_2)) \cong \mathbb{C}$$

as desired. \square

We end this section by recalling two properties of coarse moduli spaces that we will use later:

Fact 3.23. ([AV02, Section 2.2]) If $p : \mathcal{X} \rightarrow X$ is a coarse moduli space and F is a coherent sheaf on \mathcal{X} , then p_* is exact (so $H^i(\mathcal{X}, F) = H^i(X, p_*F)$) and $p_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

4 The spectral transform

In this section, we define the spectral transform which plays a key role in this paper, following Kenyon and Okounkov [KO06].

4.1 The discrete Abel map and extension of K

We will construct an extension of $K(x)$ to a G -invariant map of G -equivariant locally free sheaves on \mathbb{P}^2 via a construction of Fock [Foc15] called the *discrete Abel map*

$$\mathbf{d} : B \sqcup W \rightarrow \mathbb{Z}\langle D_0, D_1, D_2 \rangle,$$

defined as follows:

1. Let w_0 be a white vertex of Γ , which we take to be the white vertex at the bottom left of the fundamental domain. Define $\mathbf{d}(w_0) = 0$.
2. For any path γ in the fundamental domain from v_1 to v_2 ,

$$\mathbf{d}(v_2) - \mathbf{d}(v_1) = \sum_{i=0}^2 \sum_{\alpha \in Z_i} (\alpha, \gamma) D_i,$$

where (\cdot, \cdot) is the intersection index in \mathbb{T} . Note that since the path γ must be contained in the fundamental domain, this is well-defined independent of the choice of path γ .

An alternate description is as follows: Let e be an edge contained inside the fundamental domain. There are two zig-zag paths $\alpha_r(e), \alpha_l(e)$ that contain e . Let $\alpha_r(e) \in Z_i$ and $\alpha_l(e) \in Z_j$. Then we have

$$\mathbf{d}(b) = \mathbf{d}(w) + D_i + D_j.$$

Define for each black vertex b the G -equivariant line bundle $\mathcal{E}_b := \mathcal{O}_{\mathbb{P}^2}^G(\mathbf{d}(b) - D)$ and for each white vertex w the G -equivariant line bundle $\mathcal{F}_w := \mathcal{O}_{\mathbb{P}^2}^G(\mathbf{d}(w))$, where $D = D_0 + D_1 + D_2$ (see Proposition 3.11 in the Appendix for the definition). Let $\mathcal{E} := \bigoplus_{b \in B} \mathcal{E}_b$ and $\mathcal{F} := \bigoplus_{w \in W} \mathcal{F}_w$. The following Proposition is a special case of a more general result proved in [GGK22].

Proposition 4.1. *$K(x)$ extends to a G -invariant map of locally free sheaves*

$$\mathcal{E} \xrightarrow{\tilde{K}(x)} \mathcal{F}.$$

Notation 4.2. We will denote by $\tilde{K}(x)$ the extension of the Kasteleyn matrix as above.

Proof. We need to show that for each edge $e = bw$, we have

$$\begin{aligned} K(x)_{w,b} &\in \mathrm{Hom}(\mathcal{E}_b, \mathcal{F}_w)^G \\ &\cong H^0(\mathbb{P}^2, \mathcal{E}_b^\vee \otimes \mathcal{F}_w)^G. \end{aligned}$$

We have three cases:

1. The edge e does not cross either γ_1 or γ_2 : We have $\mathbf{d}(b) = \mathbf{d}(w) + D_i + D_j$. Let $k \in \{0, 1, 2\} \setminus \{i, j\}$. Then $\mathcal{E}_b = \mathcal{O}_{\mathbb{P}^2}^G(\mathbf{d}(b) - D) = \mathcal{O}_{\mathbb{P}^2}^G(\mathbf{d}(w) - D_k)$, so $\mathcal{E}_b^\vee \otimes \mathcal{F}_w \cong \mathcal{O}_{\mathbb{P}^2}^G(D_k)$. By Example 3.16, $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_k))^G = \mathbb{C}\{1\}$. Since $K_{w,b} = x(bw) \cdot 1$, we have $K_{w,b} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_k))^G$.
2. The edge e crosses γ_2 : We have

$$\begin{aligned} \mathbf{d}(b) &= \mathbf{d}(w) + (d-1)(D_2 - D_1) + (D_2 - D_3) \\ &= \mathbf{d}(w) + d(D_2 - D_1) + (D_1 + D_3) \\ &= \mathbf{d}(w) + \mathrm{div} z + D_1 + D_3, \end{aligned}$$

where we have used $\mathrm{div} z = d(D_2 - D_1)$. Therefore $\mathcal{E}_b^\vee \otimes \mathcal{F}_w \cong \mathcal{O}_{\mathbb{P}^2}^G(D_2 - \mathrm{div} z)$. By Examples 3.16 and 3.18, we have $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_2 - \mathrm{div} z))^G = \mathbb{C}\{z\}$. Since $K(x)_{w,b} = x(bw) \cdot z$, we get that $K(x)_{w,b} \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_2 - \mathrm{div} z))^G$.

3. The edge e crosses γ_2 : This is almost identical to item 2. □

In fact, we have the following stronger result that every G -invariant map $\mathcal{E} \rightarrow \mathcal{F}$ arises from the Kasteleyn matrix.

Corollary 4.3. *The map $\mathbb{C}^E \rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{F})^G$ given by $\dot{x} \mapsto \tilde{K}(\dot{x})$ is an isomorphism.*

Proof. We need to show that

$$\mathrm{Hom}(\mathcal{E}_b, \mathcal{F}_w)^G \cong \begin{cases} \mathbb{C} & \text{if } bw \text{ is an edge of } \Gamma; \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry, we can assume the w is the white vertex in the lower left corner of the fundamental domain (see Figure 1) and that $\mathbf{d}(w) = 0$. Then we have

$$\mathbf{d}(b) = \frac{\alpha}{d} + \frac{\beta}{d} + k \left(\frac{\beta}{d} - \frac{\gamma}{d} \right) + l \left(\frac{\alpha}{d} - \frac{\gamma}{d} \right) \text{ for some } k, l \in 0, \dots, d-1,$$

from which we get

$$\mathcal{H}om(\mathcal{E}_b, \mathcal{F}_w) = \mathcal{O}_{\mathbb{P}^2}^G \left(\frac{\gamma}{d} + k \left(\frac{\gamma}{d} - \frac{\beta}{d} \right) + l \left(\frac{\gamma}{d} - \frac{\alpha}{d} \right) \right).$$

Using Proposition 3.14, we get that the dimension of $\mathrm{Hom}(\mathcal{E}_b, \mathcal{F}_w)^G$ is the number of lattice points in the polygon $P(-\frac{l}{d}, -\frac{k}{d}, \frac{k+l+1}{d})$, which is the triangle

$$\mathrm{Convex-hull} \left\{ \left(\frac{k}{d}, \frac{l}{d} \right), \left(\frac{k+1}{d}, \frac{l}{d} \right), \left(\frac{k}{d}, \frac{l+1}{d} \right) \right\},$$

which contains exactly one lattice point when $(k, l) \in \{(0, 0), (d-1, 0), (0, d-1)\}$, which corresponds to the three edges, and no lattice points otherwise. \square

4.2 The spectral transform

Due to Proposition 4.1, the Kasteleyn matrix $\tilde{K}(x)$ sits in the exact sequence

$$0 \rightarrow \mathcal{E} \xrightarrow{\tilde{K}(x)} \mathcal{F} \rightarrow \mathcal{L} \rightarrow 0,$$

where $\mathcal{L} := \mathrm{coker} \tilde{K}(x)$ is a G -equivariant sheaf called the *spectral sheaf*. Since $\tilde{C} = \{\det \tilde{K}(x) = 0\}$, the sheaf \mathcal{L} is supported on \tilde{C} , and if \tilde{C} is smooth (which holds for a generic \mathbb{C}^\times -local system x), then \mathcal{L} is the pushforward of a G -equivariant line bundle L on \tilde{C} .

Proposition 4.4. *In homogeneous coordinates, the Kasteleyn matrix has the form shown in Figure 6.*

Proof. As in the proof of Proposition 4.1, we again have three cases. Let $e = bw$.

1. The edge e does not cross either γ_1 or γ_2 : We have $K_{w,b}(x) \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_k))^G$, and by Example 3.16, $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_k))^G \cong \mathbb{C}\{x_k\}$ with $K_{w,b}(x) = x(bw) \cdot 1 \mapsto x(bw) \cdot x_k$.

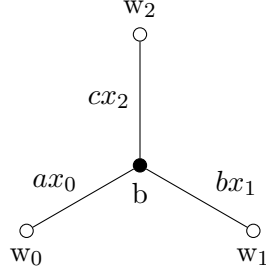


Figure 6: The Kasteleyn matrix $\tilde{K}(x)$ in homogeneous coordinates, where $a = x(bw_0)$, $b = x(bw_1)$ and $c = x(bw_2)$.

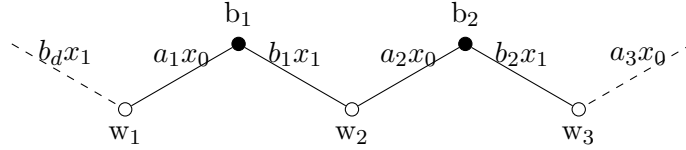


Figure 7: The Kasteleyn matrix $\tilde{K}_\alpha(x)$ of a zig-zag path (oriented from left to right), where $a_1 = x(b_1w_1)$ etc.

2. The edge e crosses γ_1 : We have $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(D_2 - \text{div } z))^G = \mathbb{C}\{z\} \cong \mathbb{C}\{x_2\}$, $z \mapsto x_2$, under which $K(x)_{w,b} = x(bw) \cdot z \mapsto x(bw) \cdot x_2$.
3. The edge e crosses γ_2 : Similar to item 2.

□

Therefore when $x_2 = 0$, the Kasteleyn matrix becomes block-diagonal with blocks corresponding to zig-zag paths with homology class $(1, 0)$.

Definition 4.5. Let α be such a zig-zag path, with Kasteleyn matrix as shown in Figure 7. The corresponding block of $\tilde{K}(x) \Big|_{x_2=0}$ is

$$\tilde{K}_\alpha(x) := \begin{bmatrix} a_1x_0 & & & & b_dx_1 \\ b_1x_1 & a_2x_0 & & & \\ & b_2x_1 & a_3x_0 & & \\ & & \ddots & \ddots & \\ & & & & \ddots \end{bmatrix}. \quad (6)$$

The matrix $\tilde{K}_\alpha(x)$ is singular at the d points where

$$\left(\frac{x_0}{x_1} \right)^d = (-1)^d \chi_\alpha, \quad x_2 = 0, \quad (7)$$

where $\chi_\alpha = \prod_{i=1}^d \frac{b_i}{a_i}$ is the monodromy around α . We denote these d points by $\tilde{\alpha}_1, \dots, \tilde{\alpha}_d$. The cokernel of $\tilde{K}_\alpha(x)$ is a G -equivariant sheaf that is a direct sum of skyscraper sheaves with stalks isomorphic to \mathbb{C} supported over $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_d\}$. Note that $C \cap D_2 = \{[(-1)^d \chi_\alpha : 1 : 0] = \pi(\tilde{\alpha}_1) : \alpha \in Z_2\}$, and analogously for $C \cap D_0$ and $C \cap D_1$.

Remark 4.6. When making a statement about Z_i or D_i , we will often only state it for Z_2 or D_2 , and the corresponding statements for $i = 0, 1$ are inferred from cyclic symmetry.

Definition 4.7. Let \mathcal{S} denote the moduli space of triples (C, S, ν) where

1. C is a degree d curve in \mathbb{P}^2 ,
2. S is a degree g effective divisor in C , and
3. $\nu = (\nu_0, \nu_1, \nu_2)$ is a triple of bijections, where $\nu_i : Z_i \xrightarrow{\sim} C \cap D_i$.

Fix a white vertex $w \in W$. The spectral transform

$$\kappa = \kappa_w : \mathcal{X} \rightarrow \mathcal{S}$$

is the rational map defined as follows: Let $[x] \in \mathcal{X}$ and let x be a representative.

1. C is the spectral curve,
2. S is the degree g effective divisor in C_0 given by the vanishing of the w -column of the adjugate matrix of $K(z, w)$, and
3. ν is defined as follows: Suppose $\alpha \in Z_2$. Then $\nu_2(\alpha) = [(-1)^d \chi_\alpha : 1 : 0]$.

Note that each of C, S, ν is invariant under changing the representative x of $[x]$, so κ is a well-defined map $\mathcal{X} \rightarrow \mathcal{S}$.

Theorem 4.8 ([Foc15]). *The spectral transform $\kappa : \mathcal{X} \rightarrow \mathcal{S}$ is birational.*

5 Some preliminary results

This section contains two subsections which collect some technical results which will be used later. The first subsection is more elementary and contains computations of some vector spaces which will be used in Sections 9 and 10. The goal of the second subsection is to prove that the Beauville Poisson structure (which will be defined in Section 6), which is defined on the codomain \mathbb{P}^2 in the covering map $\pi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$, can be lifted to the domain \mathbb{P}^2 . This subsection requires facts about stacks, so the unfamiliar reader might wish to skip it.

5.1 Computations of some spaces involving $\mathcal{L}|_D$

Let us describe the map $\mathcal{F} \rightarrow \mathcal{L}|_D$ i.e. the composition $\mathcal{F} \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_D$. We can do this separately at each point $\tilde{\alpha}_i$. We have

$$\mathcal{L}|_{\tilde{\alpha}_i} = \left(\text{coker } \tilde{K}(x) \right) \Big|_{\tilde{\alpha}_i} \cong \text{coker} \left(\tilde{K}(x) \Big|_{\tilde{\alpha}_i} \right) \cong \text{coker } \tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i},$$

where the first isomorphism is because of right-exactness of pullbacks, and the second is because $\tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i}$ is the only singular block. Now consider the map

$$\bigoplus_{b \in B \cap \alpha} \mathcal{E}_b \Big|_{\tilde{\alpha}_i} \xrightarrow{\tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i}} \bigoplus_{w \in W \cap \alpha} \mathcal{F}_w \Big|_{\tilde{\alpha}_i}.$$

Choose trivializations $\Phi_{\tilde{\alpha}_i, b} : \mathcal{E}_b \Big|_{\tilde{\alpha}_i} \cong \mathbb{C}$ and $\Psi_{\tilde{\alpha}_i, w} : \mathcal{F}_w \Big|_{\tilde{\alpha}_i} \cong \mathbb{C}$ such that $\tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i}$ becomes the matrix

$$1_\alpha^\# := \begin{bmatrix} 1 & & & -1 \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \end{bmatrix}.$$

We can think of this trivialization combinatorially as follows (Figure 8). Since $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}^d = (-1)^d \chi_\alpha$ at $\tilde{\alpha}_i$, if we think of the matrix (6) as defining a local system on α , then the monodromy of this local system around α is $(-1)^d$, which is the same as the local system defined by $1_\alpha^\#$. Therefore we see that these two local systems are isomorphic, and that any two choices of trivializations are related by scaling by \mathbb{C}^\times . Figure 8 shows a choice that is G -invariant.

Therefore, if we let $\Phi_{\tilde{\alpha}_i} := \bigoplus_{b \in B} \Phi_{\tilde{\alpha}_i, b}$ and $\Psi_{\tilde{\alpha}_i} := \bigoplus_{w \in W} \Psi_{\tilde{\alpha}_i, w}$, then the following diagram commutes

$$\begin{array}{ccccccc} \bigoplus_{b \in B \cap \alpha} \mathcal{E}_b \Big|_{\tilde{\alpha}_i} & \xrightarrow{\tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i}} & \bigoplus_{w \in W \cap \alpha} \mathcal{F}_w \Big|_{\tilde{\alpha}_i} & \longrightarrow & \text{coker } \tilde{K}_\alpha(x) \Big|_{\tilde{\alpha}_i} \cong \mathcal{L}|_{\tilde{\alpha}_i} & \longrightarrow & 0 \\ \Phi_{\tilde{\alpha}_i} \downarrow & & \Psi_{\tilde{\alpha}_i} \downarrow & & \cong \downarrow & & \\ \mathbb{C}^{B \cap \alpha} & \xrightarrow{1_\alpha^\#} & \mathbb{C}^{W \cap \alpha} & \xrightarrow{(1, \dots, 1)} & \mathbb{C} & \longrightarrow & 0 \end{array} \quad (8)$$

We will hereafter identify the spaces at the top of the diagram with the corresponding spaces at the bottom.

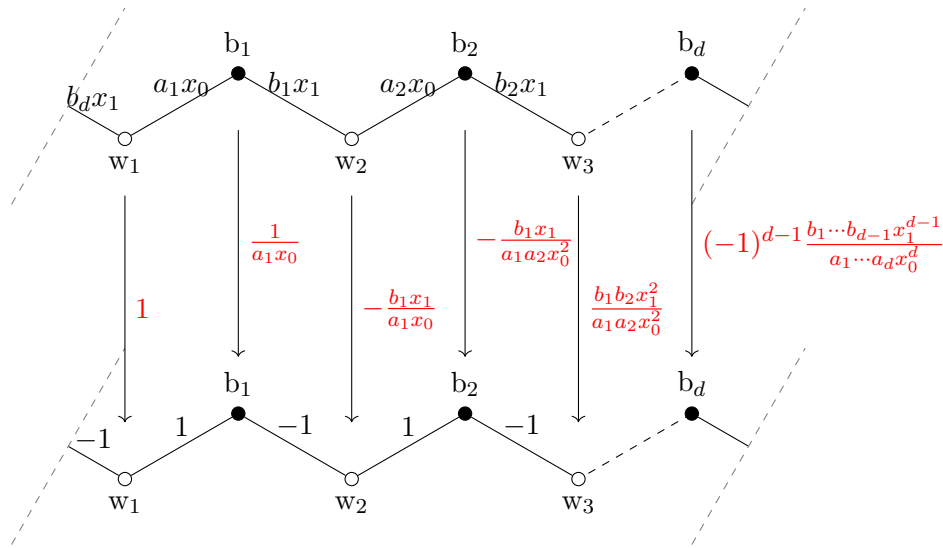


Figure 8: The trivialization (in red) at the point at infinity $\tilde{\alpha}_i$ given by $\frac{x_0}{x_1} = \zeta \chi_{\tilde{\alpha}}^{\frac{1}{d}}$ (cf. (7)), where $\zeta^d = -1$.

Lemma 5.1. *Identifying the top and bottom rows of (8), we obtain isomorphisms*

$$\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C}), \quad \mathrm{Hom}(\mathcal{F}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C}).$$

Proof. Since $\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D) \cong \bigoplus_{\alpha \in Z} \bigoplus_{i=1}^d \mathrm{Hom}(\bigoplus_{b \in B \cap \alpha} \mathcal{E}_b|_{\tilde{\alpha}_i}, \mathcal{L}|_{\tilde{\alpha}_i})$, to specify an element of $\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G$, it suffices to give the elements of $\mathrm{Hom}(\bigoplus_{b \in B \cap \alpha} \mathcal{E}_b|_{\tilde{\alpha}_i}, \mathcal{L}|_{\tilde{\alpha}_i})^G$. Using the identifications of the top row of (8) with the bottom row, we can instead describe the maps $\mathbb{C}^{B \cap \alpha} \rightarrow \mathbb{C}$ for each $\tilde{\alpha}_i$. G -invariance of the trivialization implies that the map $\mathbb{C}^{B \cap \alpha} \rightarrow \mathbb{C}$ does not depend on $\tilde{\alpha}_i$, so $\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C})$. The proof of the other isomorphism is similar. \square

Now we describe two maps that will be important for us.

Lemma 5.2. *Under the isomorphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{F})^G \cong \mathbb{C}^E$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C})$, the map $\mathrm{Hom}(\mathcal{E}, \mathcal{F})^G \rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G$ is given by*

$$\dot{x} \mapsto \left(\frac{\dot{a}_1}{a_1} - \frac{\dot{b}_d}{b_d}, \frac{\dot{a}_2}{a_2} - \frac{\dot{b}_1}{b_1}, \dots, \frac{\dot{a}_d}{a_d} - \frac{\dot{b}_{d-1}}{b_{d-1}} \right) \in \mathrm{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C}),$$

where the notation for \dot{x}, x on the zig-zag path is as in Figure 7.

Proof. We have

$$\Psi_{\tilde{\alpha}_i} \circ \tilde{K}_\alpha(\dot{x}) \Big|_{\tilde{\alpha}_i} \circ \Phi_{\tilde{\alpha}_i}^{-1} = \begin{bmatrix} \frac{\dot{a}_1}{a_1} & & & -\frac{\dot{b}_d}{b_d} \\ -\frac{\dot{b}_1}{b_1} & \frac{\dot{a}_2}{a_2} & & \\ & -\frac{\dot{b}_2}{b_2} & \frac{\dot{a}_3}{a_3} & \\ & & \ddots & \ddots \end{bmatrix},$$

so composing with $(1, 1, \dots, 1)$, we get

$$(1, 1, \dots, 1) \circ \Psi_{\tilde{\alpha}_i} \circ \tilde{K}_\alpha(\dot{x}) \Big|_{\tilde{\alpha}_i} \circ \Phi_{\tilde{\alpha}_i}^{-1} = \left(\frac{\dot{a}_1}{a_1} - \frac{\dot{b}_1}{b_1}, \frac{\dot{a}_2}{a_2} - \frac{\dot{b}_2}{b_2}, \dots, \frac{\dot{a}_d}{a_d} - \frac{\dot{b}_d}{b_d} \right).$$

□

Lemma 5.3. *Under the isomorphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{B\alpha}, \mathbb{C})$ and $\mathrm{Hom}(\mathcal{F}, \mathcal{L}|_D)^G \cong \bigoplus_{\alpha \in Z} \mathrm{Hom}(\mathbb{C}^{W\alpha}, \mathbb{C})$, the map $\mathrm{Hom}(\mathcal{F}, \mathcal{L}|_D)^G \rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G$ maps $(\phi_\alpha(w))_{w \in W\alpha} : \mathbb{C}^{W\alpha} \rightarrow \mathbb{C}$ to $(\phi_\alpha(w_1) - \phi_\alpha(w_2), \dots, \phi_\alpha(w_d) - \phi_\alpha(w_1)) : \mathbb{C}^{B\alpha} \rightarrow \mathbb{C}$.*

Proof. The image of $(\phi_\alpha(w))_{w \in W\alpha}$ is given by

$$(\phi_\alpha(w))_{w \in W\alpha} \circ 1_\alpha^\# = (\phi_\alpha(w_1) - \phi_\alpha(w_2), \dots, \phi_\alpha(w_d) - \phi_\alpha(w_1)).$$

□

5.2 $\mathrm{Ext}_{[\mathbb{P}^2/G]}^\bullet(\mathcal{L}, \mathcal{L})$

As we have seen in Section 4, the Kasteleyn matrix is naturally defined G -equivariantly on \mathbb{P}^2 , or equivalently on the stack $[\mathbb{P}^2/G]$, rather than on \mathbb{P}^2 . Therefore the goal of this subsection is to collect a few facts about computing certain Ext groups on 2-dimensional Deligne-Mumford stacks.

We start with some notation, which is only used in this section. Assume $i : \mathcal{C} \rightarrow \mathcal{X}$ is the inclusion of a stacky curve in a 2-dimensional smooth Deligne-Mumford stack \mathcal{X} , and that \mathcal{E} is a line bundle on \mathcal{C} . We will use these results only in the case where $\mathcal{X} = [\mathbb{P}^2/G]$ and \mathcal{C} is the spectral curve. We begin with the following:

Proposition 5.4. *Let \mathcal{N} be the normal bundle of \mathcal{C} in \mathcal{X} . Then the two complexes $Li^*i_*\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{N}^\vee[1] \oplus \mathcal{E}[0]$ are isomorphic.*

In Proposition 5.4, the numbers in square brackets [1] and [0] indicate the shift. So for example if F is a sheaf, the complex $F[i]$ is the complex with all the terms 0, except for F in degree $-i$ so that $\mathcal{H}^i(F[j]) = \mathcal{H}^{i+j}(F)$.

Proof. The proof is divided into three steps.

Step 1: $Li^*i_*\mathcal{O}_{\mathcal{C}} \cong \mathcal{N}^\vee[1] \oplus \mathcal{O}_{\mathcal{C}}[0]$.

Indeed, by definition, to construct $Li^*\mathcal{O}_{\mathcal{C}}$ it suffices to pull back a free resolution of the sheaf $i_*\mathcal{O}_{\mathcal{C}}$. We consider the free resolution $[\mathcal{O}_{\mathcal{X}}(-\mathcal{C}) \xrightarrow{f} \mathcal{O}_{\mathcal{X}}]$, where f is the equation of the curve \mathcal{C} . Its pull back to \mathcal{C} is $[i^*\mathcal{O}_{\mathcal{X}}(-\mathcal{C}) \xrightarrow{0} i^*\mathcal{O}_{\mathcal{X}}]$, and we have $\mathcal{N}^\vee = i^*\mathcal{O}_{\mathcal{X}}(-\mathcal{C})$ and $\mathcal{O}_{\mathcal{C}} = i^*\mathcal{O}_{\mathcal{X}}$.

Step 2: The two complexes $Li^*i_*\mathcal{E}$ and $(\mathcal{L} \otimes \mathcal{N}^\vee)[1] \oplus \mathcal{L}[0]$ have the same cohomology. Indeed, if F^\bullet is a complex of coherent sheaves on \mathcal{C} , then for every j we have

$$\mathcal{H}^j(F^\bullet) = i_*\mathcal{H}^j(F^\bullet) = \mathcal{H}^j(i_*F^\bullet)$$

since i_* is exact for closed immersions. Therefore it suffices to understand the cohomology of $i_*Li^*i_*\mathcal{E}$, and by the projection formula [Sta22, Tag 08EU],

$$i_*Li^*i_*\mathcal{E} = i_*(Li^*i_*(\mathcal{E}) \otimes^L \mathcal{O}_{\mathcal{C}}) = i_*\mathcal{E} \otimes^L i_*\mathcal{O}_{\mathcal{C}}.$$

Similarly, $i_*(\mathcal{E} \otimes^L Li^*i_*\mathcal{O}_{\mathcal{C}}) = i_*\mathcal{E} \otimes^L i_*\mathcal{O}_{\mathcal{C}}$ so

$$i_*(\mathcal{E} \otimes^L Li^*i_*\mathcal{O}_{\mathcal{C}}) = i_*Li^*i_*\mathcal{E}.$$

Therefore it suffices to understand the cohomology of $i_*(\mathcal{E} \otimes^L Li^*i_*\mathcal{O}_{\mathcal{C}})$. As \mathcal{E} is a line bundle and $Li^*i_*\mathcal{O}_{\mathcal{C}} \cong \mathcal{N}^\vee[1] \oplus \mathcal{O}_{\mathcal{C}}[0]$, we have $\mathcal{E} \otimes^L Li^*i_*\mathcal{O}_{\mathcal{C}} \cong (\mathcal{E} \otimes \mathcal{N}^\vee)[1] \oplus \mathcal{E}[0]$ so its cohomology in degree -1 is $(\mathcal{E} \otimes \mathcal{N}^\vee)$ whereas in degree 0 it is \mathcal{E} as desired.

Step 3: $Li^*i_*\mathcal{E} \cong (\mathcal{E} \otimes \mathcal{N}^\vee)[1] \oplus \mathcal{E}[0]$.

Recall that given a complex C^\bullet in degrees -1 and 0 , we have $C^\bullet \cong \mathcal{H}^0(C^\bullet) \oplus \mathcal{H}^{-1}(C^\bullet)[1]$ if and only if

$$\mathrm{Ext}^2(\mathcal{H}^0(C^\bullet), \mathcal{H}^{-1}(C^\bullet)) = 0.$$

Indeed, there is a triangle

$$C^\bullet \rightarrow \mathcal{H}^0(C^\bullet) \rightarrow M^\bullet \xrightarrow{+1}$$

where M^\bullet is its mapping cone, and the data of C^\bullet is equivalent to the data of the map $\mathcal{H}^0(C^\bullet) \rightarrow M^\bullet$. However, computing the long exact sequence associated to this triangle, we see that $\mathcal{H}^i(M^\bullet) = 0$ unless $i = -2$, in which case $\mathcal{H}^{-2}(M^\bullet) = \mathcal{H}^{-1}(C^\bullet)$. So the data of C^\bullet is equivalent to the data of a map in

$$\mathrm{Hom}(\mathcal{H}^0(C^\bullet), \mathcal{H}^{-1}(C^\bullet)[2]) = \mathrm{Ext}^2(\mathcal{H}^0(C^\bullet), \mathcal{H}^{-1}(C^\bullet)).$$

But $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{N}^\vee) = H^2(\mathcal{C}, \mathcal{N}^\vee) = 0$ since \mathcal{C} is a 1-dimensional Deligne-Mumford stack. \square

Lemma 5.5. *With the notations of Proposition 5.4, let \mathcal{F} be a line bundle on \mathcal{C} . Then for $i \geq 0$ we have*

$$\mathrm{Ext}_{\mathcal{X}}^i(i_*\mathcal{E}, i_*\mathcal{F}) \cong H^{i-1}(\mathcal{C}, \mathcal{N} \otimes \mathcal{E}^\vee \otimes \mathcal{F}) \oplus H^i(\mathcal{C}, \mathcal{E}^\vee \otimes \mathcal{F}).$$

Proof. We use Proposition 5.4 and adjunction between i_* and Li^* :

$$\begin{aligned} \text{Ext}^i(i_*\mathcal{E}, i_*\mathcal{F}) &= \text{Ext}^i(Li^*i_*\mathcal{E}, \mathcal{F}) = \text{Ext}^i((\mathcal{E} \otimes \mathcal{N}^\vee)[1] \oplus \mathcal{L}[0], \mathcal{F}) = \\ &= \text{Ext}^{i-1}(\mathcal{E} \otimes \mathcal{N}^\vee, \mathcal{F}) \oplus \text{Ext}^i(\mathcal{E}, \mathcal{F}) \cong H^{i-1}(\mathcal{C}, \mathcal{N} \otimes \mathcal{E}^\vee \otimes \mathcal{F}) \oplus H^i(\mathcal{C}, \mathcal{E}^\vee \otimes \mathcal{F}). \end{aligned}$$

The last step follows from the fact that \mathcal{E} and \mathcal{N} are line bundles. \square

Proposition 5.6. *Let \mathcal{X} be a smooth irreducible stacky surface and $i : \mathcal{C} \rightarrow \mathcal{X}$ a smooth stacky curve. Let $p : \mathcal{X} \rightarrow X$ be the coarse moduli space map, let $\pi : \mathcal{C} \rightarrow C$ be the coarse moduli space of \mathcal{C} and assume that X is smooth and $j : C \rightarrow X$ is a closed embedding. Assume also that p and π are generically an isomorphism. If we denote by $I_{\mathcal{C}}$ (resp. I_C) the ideal sheaf of \mathcal{C} in \mathcal{X} (resp. C in X), then $p^*I_C \cong I_{\mathcal{C}}$. Similarly, if we denote by N the normal bundle of C in X and by \mathcal{N} the one of \mathcal{C} in \mathcal{X} , then $\pi^*N = \mathcal{N}$.*

Before we prove Proposition 5.6, we observe that all the assumptions hold in the situation we will be interested on. Indeed, for us the coarse moduli space of $[\mathbb{P}^2/G]$ is the projective space \mathbb{P}^2 , the coarse moduli space map $[\mathbb{P}^2/G] \rightarrow \mathbb{P}^2$ is an isomorphism generically (as G acts faithfully on an open dense subset of \mathbb{P}^2). As in our case \mathcal{C} is generic, it intersects the locus where G has no fixed points, so also π is generically an isomorphism. Moreover as \mathcal{C} is normal, also C is normal as the coarse moduli space of a normal DM stack is normal.

Proof. Let $0 \rightarrow I_C \rightarrow \mathcal{O}_X$ be the inclusion of the ideal sheaf of C , and consider its pull-back

$$p^*I_C \rightarrow p^*\mathcal{O}_X = \mathcal{O}_{\mathcal{X}}. \quad (9)$$

As X is smooth, C is a divisor in X so I_C is Cartier. Then I_C is a line bundle, so also p^*I_C is a line bundle. Then the map (9) is injective, since a map of line bundles is injective if and only if it is injective generically, and generically p and π are isomorphisms. Therefore p^*I_C is an ideal sheaf in $\mathcal{O}_{\mathcal{X}}$, and it is clear that it is contained in $I_{\mathcal{C}}$, the ideal sheaf of \mathcal{C} . But then p^*I_C and $I_{\mathcal{C}}$ are two line bundles that agree away from \mathcal{C} , and as π is an isomorphism generically they also agree on the generic point of \mathcal{C} in \mathcal{X} . Then they agree on codimension one. But on a normal Deligne-Mumford stack, two line bundles that agree in codimension one are isomorphic. Since \mathcal{X} is smooth, $I_{\mathcal{C}} = p^*I_C$. Now the desired statement follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{X} \\ \downarrow \pi & & \downarrow p \\ C & \xrightarrow{j} & X \end{array}$$

and the fact that $N = j^*I_C^\vee$ (resp. $\mathcal{N} = i^*I_{\mathcal{C}}^\vee$). \square

Corollary 5.7. *Assume that we are in the situation of Proposition 5.6, let \mathcal{E} be a line bundle on \mathcal{X} and $E := \pi_*\mathcal{E}$. Assume that E is a line bundle as well. Then $\pi_*i^*\mathcal{E} \cong j^*p_*\mathcal{E}$.*

Proof. Consider the exact sequence $0 \rightarrow I_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow i_*\mathcal{O}_{\mathcal{G}} \rightarrow 0$. Since \mathcal{E} is a line bundle, such a sequence remains exact if we tensor it with \mathcal{E} . Similarly, as p_* is exact, also the following sequence is exact:

$$0 \rightarrow p_*(\mathcal{E} \otimes I_{\mathcal{G}}) \rightarrow p_*\mathcal{E} \rightarrow p_*(\mathcal{E} \otimes i_*\mathcal{O}_{\mathcal{G}}) \rightarrow 0.$$

Now, from Proposition 5.6, $I_{\mathcal{G}} = p^*I_C$ so $p_*(\mathcal{E} \otimes I_{\mathcal{G}}) \cong p_*(\mathcal{E}) \otimes I_C$. Therefore we can tensor the exact sequence $0 \rightarrow I_C \rightarrow \mathcal{O}_X \rightarrow j_*\mathcal{O}_C \rightarrow 0$ by the line bundle $p_*(\mathcal{E})$ to get

$$0 \rightarrow I_C \otimes p_*(\mathcal{E}) \rightarrow p_*(\mathcal{E}) \rightarrow p_*(\mathcal{E}) \otimes j_*\mathcal{O}_C \rightarrow 0.$$

In particular $p_*(\mathcal{E}) \otimes j_*\mathcal{O}_C \cong p_*(\mathcal{E} \otimes i_*\mathcal{O}_{\mathcal{G}})$, as they are the cokernel of the same morphism. But by projection formula $p_*(\mathcal{E}) \otimes j_*\mathcal{O}_C \cong j_*j^*p_*(\mathcal{E})$ and $p_*(\mathcal{E} \otimes i_*\mathcal{O}_{\mathcal{G}}) \cong p_*i_*i^*\mathcal{E} \cong j_*\pi^*i^*\mathcal{E}$. Therefore $j_*\pi^*i^*\mathcal{E} \cong j_*j^*p_*(\mathcal{E})$. The desired statement follows since j_* is fully faithful. \square

We now specialize to the case $\mathcal{X} = [\mathbb{P}^2/G]$ and $i : \mathcal{C} \rightarrow [\mathbb{P}^2/G]$ the inclusion of a generic spectral curve, with coarse moduli space maps $p : [\mathbb{P}^2/G] \rightarrow \mathbb{P}^2$ and $\pi : \mathcal{C} \rightarrow C$. As usual, we denote by $j : C \rightarrow \mathbb{P}^2$ the closed embedding induced by i on the level of coarse moduli spaces. The last result of this section is the following

Proposition 5.8. *Let \mathcal{E} be a line bundle on \mathcal{C} . Then the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ext}_{[\mathbb{P}^2/G]}^1(i_*\mathcal{E}, i_*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2)) & \xrightarrow{\theta^{\frac{1}{2}}} & \mathrm{Ext}_{[\mathbb{P}^2/G]}^1(i_*\mathcal{E}, i_*\mathcal{E}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}_{\mathbb{P}^2}^1(j_*E, j_*E \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) & \xrightarrow{\theta} & \mathrm{Ext}_{\mathbb{P}^2}^1(j_*E, j_*E) \end{array} .$$

where the horizontal maps are the canonical maps given by $i_*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2) \rightarrow i_*\mathcal{E}$ and $i_*E \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow i_*E$, and the vertical ones are isomorphisms.

Proof. The argument proceeds as follows. First, using Lemma 5.5, we will pass from the Ext groups above to an analogous diagram between cohomologies of line bundles on \mathcal{C} and C . Then we will conclude using Fact 3.23 and the description of π_* and p_* for certain line bundles on \mathcal{C} and $[\mathbb{P}^2/G]$.

Observe first that, using Fact 3.23 and Proposition 5.6, we have

$$\pi_*\mathcal{N} \cong \pi_*(\pi^*N \otimes \mathcal{O}_{\mathcal{C}}) \cong N \otimes \pi_*\mathcal{O}_{\mathcal{C}} \cong N \text{ and } \pi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_C. \quad (10)$$

Moreover, from Corollary 5.7 and Proposition 3.22, $\pi_* i^* \mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2) \cong j^* \mathcal{O}_{\mathbb{P}^2}(-3)$. Therefore replacing $\mathcal{O}_{\mathcal{C}}$ in (10) with $i^* \mathcal{O}_{\mathbb{P}^2}^G(-D_0 - D_1 - D_2)$, which we will denote by $i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)$, we have

$$\pi_*(\mathcal{N} \otimes i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) \cong N \otimes j^* \mathcal{O}_{\mathbb{P}^2}(-3) \quad \text{and} \quad \pi_* i^* \mathcal{O}_{\mathbb{P}^2}^G(-D) \cong j^* \mathcal{O}_{\mathbb{P}^2}(-3). \quad (11)$$

Now, from Lemma 5.5 we have

$$\begin{aligned} \text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}^G(-D)) &= H^0(\mathcal{C}, \mathcal{N} \otimes i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) \oplus H^1(\mathcal{C}, i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)), \quad \text{and} \\ \text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E}) &= H^0(\mathcal{C}, \mathcal{N}) \oplus H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}). \end{aligned}$$

So from Fact 3.23 and the equalities (10) and (11), we get

$$\begin{aligned} H^0(\mathcal{C}, \mathcal{N} \otimes i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) \oplus H^1(\mathcal{C}, i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) &= H^0(C, N \otimes j^* \mathcal{O}_{\mathbb{P}^2}(-3)) \oplus H^1(C, j^* \mathcal{O}_{\mathbb{P}^2}(-3)), \quad \text{and} \\ H^0(\mathcal{C}, \mathcal{N}) \oplus H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) &= H^0(C, N) \oplus H^1(C, \mathcal{O}_C). \end{aligned}$$

Therefore using again Lemma 5.5 we get

$$\text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E} \otimes i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) \cong \text{Ext}_{\mathbb{P}^2}^1(j_* E, j_* E \otimes i^* \mathcal{O}_{\mathbb{P}^2}(-3)) \quad \text{and} \quad \text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E}) \cong \text{Ext}_{\mathbb{P}^2}^1(j_* E, j_* E).$$

Finally to get the desired commutativity it suffices to observe that the natural maps

$$\text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}^G(-D)) \rightarrow \text{Ext}_{\mathbb{P}^2/G}^1(i_* \mathcal{E}, i_* \mathcal{E}) \quad \text{and} \quad \text{Ext}_{\mathbb{P}^2}^1(j_* E, j_* E \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \rightarrow \text{Ext}_{\mathbb{P}^2}^1(j_* E, j_* E)$$

are induced, under the isomorphism of Lemma 5.5, by the natural maps

$$\begin{aligned} H^0(\mathcal{C}, \mathcal{N} \otimes i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) \oplus H^1(\mathcal{C}, i^* \mathcal{O}_{\mathbb{P}^2}^G(-D)) &\rightarrow H^0(\mathcal{C}, \mathcal{N}) \oplus H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \\ H^0(C, N(-3)) \oplus H^1(C, \mathcal{O}_C(-3)) &\rightarrow H^0(C, N) \oplus H^1(C, \mathcal{O}_C) \end{aligned}$$

given by the inclusion of the ideal sheaf of D and the three lines at infinity in \mathbb{P}^2 respectively. \square

6 A cover of the Beauville integrable system

Consider the Poisson structure $\theta = z \frac{\partial}{\partial z} \wedge w \frac{\partial}{\partial w} \in H^0(\mathbb{P}^2, \wedge^2 T\mathbb{P}^2)$ on \mathbb{P}^2 . Given a generic collection of $3d - 1$ points in $\text{Sym}^d D_0 \times \text{Sym}^d D_1 \times \text{Sym}^{d-1} D_2$, there is a g dimensional space of degree d curves through them. Any such curve passes through one more point in D_2 , giving us a collection of $3d$ points. Let $\mathcal{Y} \subset \text{Sym}^d D_0 \times \text{Sym}^d D_1 \times \text{Sym}^d D_2$ denote the subset of $3d$ points that arise in this way. In other words, there is a unique relation among the $3d$ points of intersection of a generic degree d curve with D , and \mathcal{Y} is the subset satisfying this relation.

We define the rational map $\rho : \mathcal{S} \dashrightarrow \text{Sym}^g \mathbb{P}^2 \times \mathcal{Y}$ mapping (C, S, ν) to (S, Y) where $Y = (C \cap D_0, C \cap D_1, C \cap D_2)$. Since the curve C is recovered from (S, Y) as the unique

curve through the points of S and Y , the map ρ is generically a d^3 -fold covering, with covering group $S_d \times S_d \times S_d$ permuting the bijections ν . Therefore the tangent map

$$d\rho : T_{(C,S,\nu)}\mathcal{S} \rightarrow T_{(S,Y)}\mathrm{Sym}^g\mathbb{P}^2 \times \mathcal{Y}$$

is an isomorphism. The tangent space to $\mathrm{Sym}^g\mathbb{P}^2$ at $S = p_1 + \cdots + p_g$ is given by

$$T_S\mathrm{Sym}^g\mathbb{P}^2 \cong T_{p_1}\mathbb{P}^2 \oplus \cdots \oplus T_{p_g}\mathbb{P}^2,$$

and the cotangent space is

$$T_S^*\mathrm{Sym}^g\mathbb{P}^2 \cong T_{p_1}^*\mathbb{P}^2 \oplus \cdots \oplus T_{p_g}^*\mathbb{P}^2,$$

and therefore θ defines a Poisson structure on \mathcal{S} , given under the isomorphism $d\rho$ by the direct sum of the maps

$$\theta_{p_i} : T_{p_i}^*\mathbb{P}^2 \rightarrow T_{p_i}\mathbb{P}^2,$$

and the zero map from $T_Y^*\mathcal{Y} \rightarrow T_Y\mathcal{Y}$.

Let \mathcal{B} denote the space of degree d curves in \mathbb{P}^2 , and let $\tilde{\mathcal{B}}$ denote the space of pairs (C, ν) , where $C \in \mathcal{B}$ and ν is as in the definition of \mathcal{S} . Then we have fibrations $\mathcal{S} \rightarrow \mathcal{B}$ and $\mathrm{Sym}^g\mathbb{P}^2 \times \mathcal{Y} \rightarrow \mathcal{B}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\rho} & \mathrm{Sym}^g\mathbb{P}^2 \times \mathcal{Y} \\ \downarrow (C,S,\nu) \mapsto (C,\nu) & & \downarrow \\ \tilde{\mathcal{B}} & \xrightarrow{(C,\nu) \mapsto C} & \mathcal{B} \end{array}$$

Let \mathcal{S}_Y (resp. $\mathcal{B}_Y, \tilde{\mathcal{B}}_Y$) denote the subspace of \mathcal{S} (resp. $\mathcal{B}, \tilde{\mathcal{B}}$) such that C contains the points of Y . Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{S}_Y & \xrightarrow{\rho|_{\mathcal{S}_Y}} & \mathrm{Sym}^g\mathbb{P}^2 \\ \downarrow & & \downarrow \\ \tilde{\mathcal{B}}_Y & \xrightarrow{(C,\nu) \mapsto C} & \mathcal{B}_Y \end{array},$$

Theorem 6.1 ([Bea91, Proposition 3]). *The fibration $\mathrm{Sym}^g\mathbb{P}^2 \dashrightarrow \mathcal{B}_Y$ is Lagrangian, so we have an algebraic integrable system on a dense open subset of $\mathrm{Sym}^g\mathbb{P}^2$.*

It follows from Theorem 6.1 that the fibration $\mathcal{S}_Y \rightarrow \tilde{\mathcal{B}}_Y$ is Lagrangian, and therefore give algebraic integrable systems that are d^3 -fold covers of the Beauville integrable system.

7 The main theorem

Suppose $[x] \in \mathcal{X}_\chi$. As we have seen in Section 4.2, the points $D \cap C$ of intersection of the spectral curve C with D are determined by χ , i.e. χ determines a point $Y \in \mathcal{Y}$, and therefore the restriction of the spectral transform

$$\kappa : \mathcal{X}_\chi \dashrightarrow \mathcal{S}_Y$$

is well-defined and birational. Moreover, since C is determined by $C \cap D$ and the coefficients $H_{i,j}$ for $(i,j) \in N^\circ \cap \mathbb{Z}^2$, and ν is determined by χ , we obtain a birational map

$$\mathcal{U} \dashrightarrow \tilde{\mathcal{B}}_Y$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}_\chi & \dashrightarrow^{\kappa} & \mathcal{S}_Y \\ \downarrow & & \downarrow \\ \mathcal{U} & \dashrightarrow & \tilde{\mathcal{B}}_Y \end{array} . \quad (12)$$

Our main theorem is:

Theorem 7.1. *The commutative diagram (12) is a birational isomorphism of integrable systems.*

7.1 Plan of proof

The only thing to check is that κ is Poisson.

Fix d , and let $g = \binom{d-1}{2}$ denote the genus of a degree d curve in \mathbb{P}^2 . Consider the moduli space \mathcal{N} of pairs (i_*L, ν) where L is a degree g line bundle on a curve C of degree d in \mathbb{P}^2 , $i : C \hookrightarrow \mathbb{P}^2$ is the embedding and ν is as in the definition of \mathcal{S} . We define a rational map

$$\begin{aligned} \psi : \mathcal{S} &\dashrightarrow \mathcal{N} \\ (C, S, \nu) &\mapsto (i_*\mathcal{O}_C(S), \nu). \end{aligned}$$

Let $\mathcal{M}(d)$ denote the moduli space of stable sheaves \mathcal{H} on \mathbb{P}^2 with Hilbert polynomial

$$h(n) = dn + 1.$$

The moduli space $\mathcal{M}(d)$ has a dense open subset of sheaves of the form $\mathcal{H} = i_*L$, where

1. C is a nonsingular curve of degree d in \mathbb{P}^2 , with embedding $i : C \hookrightarrow \mathbb{P}^2$, and

2. L is a line bundle of degree g on C .

By deformation theory, the tangent space to $\mathcal{M}(d)$ at $[\mathcal{H}]$ is canonically isomorphic to $\text{Ext}^1(\mathcal{H}, \mathcal{H})$, and by Grothendieck-Serre duality, the cotangent space is canonically isomorphic to $\text{Ext}^1(\mathcal{H}, \mathcal{H}(-D))$. Bottacin [Bot95], generalizing earlier work of Mukai [Muk84] and Tyurin [Tyu88], proved that

$$T_{[\mathcal{H}]}^* \mathcal{M}(d) \cong \text{Ext}^1(\mathcal{H}, \mathcal{H}(-D)) \xrightarrow{\theta} \text{Ext}^1(\mathcal{H}, \mathcal{H}) \cong T_{[\mathcal{H}]} \mathcal{M}(d)$$

is a Poisson structure. Since ν is discrete, the tangent space to \mathcal{N} at $[(i_*L, \nu)]$ is isomorphic to $T_{[i_*L]} \mathcal{M}(d)$, and therefore we get an induced Poisson structure on \mathcal{N} .

Proposition 7.2. [HK00, Proposition 3.18] *The birational map ψ is Poisson.*

Let $[x] \in \mathcal{X}$, $(C, S, \nu) = \kappa(x)$, $\mathcal{H} = i_* \mathcal{O}_C(S)$ and $\mathcal{L} = \text{coker } \tilde{K}(x)$. Let $\tilde{i} : \tilde{C} \hookrightarrow \mathbb{P}^2$ denote the embedding. By [KO06, Lemma 1], there is a divisor \tilde{S}_∞ of degree $\frac{3d^2(d-1)}{2}$ supported over the points at infinity of \tilde{C} (i.e. on $\tilde{C} \cap D$) such that

$$\begin{aligned} \mathcal{L} &= \pi^* \mathcal{H} \otimes \tilde{i}_* \mathcal{O}_{\tilde{C}}(\tilde{S}_\infty) \\ &= \tilde{i}_* \mathcal{O}_{\tilde{C}}(\pi^{-1}(S) + S_\infty). \end{aligned}$$

Let $\tilde{L} := \mathcal{O}_{\tilde{C}}(\pi^{-1}(S) + \tilde{S}_\infty)$ denote the line bundle over \tilde{C} , so $\mathcal{L} = \tilde{i}_* \tilde{L}$. Then from the isomorphisms in Proposition 5.8 we have

$$\text{Ext}^1(\pi^* \mathcal{H}, \pi^* \mathcal{H})^G = \text{Ext}^1(\mathcal{H}, \mathcal{H}),$$

and using Lemma 5.5 (twice) gives

$$\text{Ext}^1(\pi^* \mathcal{H}, \pi^* \mathcal{H})^G \cong \text{Ext}^1(\mathcal{L}, \mathcal{L})^G,$$

so that we have

$$T_{(C, S, \nu)} \mathcal{S} \cong \text{Ext}^1(\mathcal{H}, \mathcal{H}) \cong \text{Ext}^1(\pi^* \mathcal{H}, \pi^* \mathcal{H})^G \cong \text{Ext}^1(\mathcal{L}, \mathcal{L})^G. \quad (13)$$

Note that by definition $\mathcal{O}_{[\mathbb{P}^2/G]}(-D/d)$ is the G -equivariant line bundle $\mathcal{O}_{\mathbb{P}^2}^G(-D)$, and the section $\theta^{\frac{1}{d}}$ is the G -invariant section θ . Therefore to prove Theorem 7.1, by Proposition 5.8, it suffices to show that the following diagram commutes:

$$\begin{array}{ccccc} T_{[x]}^* \mathcal{X} & \xrightarrow[\cong]{\text{Poincaré duality}} & \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) & \xrightarrow{\text{pr}_1} & T_{[x]} \mathcal{X} = \text{Loc}(\Gamma, \mathbb{C}) \\ \uparrow d\kappa^* & & & & \downarrow d\kappa \\ T_{(C, S, \nu)}^* \mathcal{S} & \xrightarrow[\cong]{\text{Grothendieck-Serre duality}} & \text{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G & \xrightarrow{\theta} & T_{(C, S, \nu)} \mathcal{S} \cong \text{Ext}^1(\mathcal{L}, \mathcal{L})^G \end{array} \quad (14)$$

which we will do by computing all the maps explicitly.

8 Computation of $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$ and $d\kappa$

In this section we will compute $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$ using the Kasteleyn resolution of \mathcal{L} .

Suppose $(C, S, \nu) \in \mathcal{S}$ is generic and $\tilde{L} := \mathcal{O}_{\tilde{C}}(\pi^{-1}(S) + \tilde{S}_\infty)$. Then by Theorem 4.8, \mathcal{L} is the cokernel of the injective map $\mathcal{E} \xrightarrow{\tilde{K}(x)} \mathcal{F}$.

Notation 8.1. We denote by K^\bullet the complex in degrees -1 and 0 given by $K^\bullet := \left[\mathcal{E} \xrightarrow{\tilde{K}(x)} \mathcal{F} \right]$.

To compute $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$, we use the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}^2, \mathcal{E}xt^q(\mathcal{L}, \mathcal{L}))^G \Rightarrow \text{Ext}^{p+q}(\mathcal{L}, \mathcal{L})^G. \quad (15)$$

The cohomology groups $H^\bullet(\mathbb{P}^2, \cdot)^G$ can be computed in the same way as Čech cohomology using a G -invariant open cover [Gro57, Section V]. In our case, standard affine open cover of \mathbb{P}^2 , $\mathcal{U} = (U_i)_{i \in \{0,1,2\}}$ where $U_i = \{[x_0 : x_1 : x_2] : x_i \neq 0\}$, is G -invariant.

We recall the definition of the $\mathcal{H}om$ -complex (cf. [HL10, Section 10.1.1]). Suppose E^\bullet and F^\bullet are finite complexes of locally free sheaves, with differentials d_E and d_F respectively. The complex $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$ is defined as

$$\mathcal{H}om^n(E^\bullet, F^\bullet) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}om(E^i, F^{i+n}),$$

with differential $d(f) = d_F \circ f - (-1)^{\deg(f)} f \circ d_E$. Therefore complex $\mathcal{H}om^\bullet(K^\bullet, K^\bullet)$ is

$$\left[\mathcal{H}om(\mathcal{F}, \mathcal{E}) \xrightarrow{d^{-1}} \mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}) \xrightarrow{d^0} \mathcal{H}om(\mathcal{E}, \mathcal{F}) \right],$$

where the differentials are $d^{-1}(f) = (f\tilde{K}(x), \tilde{K}(x)f)$ and $d^0(f, g) = \tilde{K}(x)f - g\tilde{K}(x)$.

The 0-th page of the spectral sequence (15) is

$$E_0^{p,q} = \check{C}^p(\mathcal{H}om^q(K^\bullet, K^\bullet))^G,$$

with anticommuting differentials

$$\begin{array}{c} \check{C}^p(\mathcal{H}om^{q+1}(K^\bullet, K^\bullet))^G \\ \delta_{\uparrow}^{p,q} = (-1)^p d^q \uparrow \\ \check{C}^p(\mathcal{H}om^q(K^\bullet, K^\bullet))^G \xrightarrow{\delta_{\rightarrow}^{p,q} = \check{d}^p} \check{C}^{p+1}(\mathcal{H}om^q(K^\bullet, K^\bullet))^G \end{array},$$

where \check{d} is the Čech differential. The nontrivial portion of this double complex is:

$$\begin{array}{ccccc}
\check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G \\
\uparrow & & \uparrow & & \uparrow \\
\check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G \\
\uparrow & & \uparrow & & \uparrow \\
\check{C}^0(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G
\end{array} \tag{16}$$

Therefore a 1-cocycle for the hypercohomology $\mathbb{H}^1(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet))^G)$ is a triple (l_i, m_{ij}, n_{ijk}) , where

$$l_i \in \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G, \quad m_{ij} \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G, \quad n_{ijk} \in \check{C}^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G,$$

satisfying the cocycle equations (for the differential $\delta_{\rightarrow} + \delta_{\uparrow}$)

$$l_j - l_i - (\tilde{K}(x)m_{ij} - m_{ij}\tilde{K}(x)), \quad m_{ij} - m_{ik} + m_{jk} + (n_{ijk}\tilde{K}(x), \tilde{K}(x)n_{ijk}) = 0.$$

Proposition 8.2. *The two complexes of vector spaces*

$$R\mathrm{Hom}(\mathcal{L}, \mathcal{L})^G \text{ and } \left[\frac{\mathrm{Hom}(\mathcal{E}, \mathcal{E})^G \oplus \mathrm{Hom}(\mathcal{F}, \mathcal{F})^G}{\substack{(f,g) \mapsto \tilde{K}(x)f - g\tilde{K}(x)}}} \rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{F})^G \right] \tag{17}$$

are quasi-isomorphic.

Proof. We can compute $\mathbb{H}^1(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet))^G)$ using the spectral sequence, but in the opposite order compared to (15), so we first compute the cohomology of the double complex horizontally, then vertically and so on. Computing G -invariant Čech cohomology, we get the E_1 page:

$$\begin{array}{ccccc}
H^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G & \longrightarrow & H^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G & \longrightarrow & H^2(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G \\
\uparrow & & \uparrow & & \uparrow \\
H^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G & \longrightarrow & H^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G & \longrightarrow & H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G \\
\uparrow & & \uparrow & & \uparrow \\
H^0(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & H^1(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G
\end{array} \tag{18}$$

Since line bundles on \mathbb{P}^2 have no H^1 , the H^1 -column vanishes. Since forgetting the G -equivariant structure, \mathcal{E}_b is $\mathcal{O}_{\mathbb{P}^2}(-1)$ and \mathcal{F}_w is $\mathcal{O}_{\mathbb{P}^2}$, we get that the H^2 -column also van-

ishes, and similarly $H^0(\mathcal{H}om(\mathcal{F}, \mathcal{E})) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$. Therefore the complex is:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \underline{H^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G} & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

so the E_2 page has only two nonzero entries (at $E_2^{0,0}$ and $E_2^{0,1}$), and the spectral sequence has already converged. The entry $E_2^{0,1}$ is the vector space on the right hand side of (17). \square

Using Corollary 4.3 and the previous result, we have:

Corollary 8.3. *A representative for an element of $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$ is given by an element of $\mathbb{C}^E \cong \text{Hom}(\mathcal{E}, \mathcal{F})^G$.*

Using Examples 3.16, 3.17 and 3.18, we get the following three cases

$$\begin{aligned}
\text{Hom}(\mathcal{E}_b, \mathcal{F}_w)^G &\cong \begin{cases} \mathbb{C} & \text{if } bw \text{ is an edge of } \Gamma; \\ 0 & \text{otherwise,} \end{cases} & \text{Hom}(\mathcal{E}_b, \mathcal{E}_{b'})^G &\cong \begin{cases} \mathbb{C} & \text{if } b = b'; \\ 0 & \text{otherwise,} \end{cases} & \text{and} \\
\text{Hom}(\mathcal{F}_w, \mathcal{F}_{w'})^G &\cong \begin{cases} \mathbb{C} & \text{if } w = w'; \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{Hom}(\mathcal{E}, \mathcal{F})^G &= \bigoplus_{b \in B} \bigoplus_{w \in W} \text{Hom}(\mathcal{E}_b, \mathcal{F}_w)^G \cong \mathbb{C}^E, \\
\text{Hom}(\mathcal{E}, \mathcal{E})^G &= \bigoplus_{b \in B} \bigoplus_{b' \in B} \text{Hom}(\mathcal{E}_b, \mathcal{E}_{b'})^G \cong \mathbb{C}^B, \\
\text{Hom}(\mathcal{F}, \mathcal{F})^G &= \bigoplus_{w \in W} \bigoplus_{w' \in W} \text{Hom}(\mathcal{F}_w, \mathcal{F}_{w'})^G \cong \mathbb{C}^W.
\end{aligned}$$

With these isomorphisms, the complex (17) becomes $[\mathbb{C}^B \oplus \mathbb{C}^W \rightarrow \mathbb{C}^E]$, with differential $(f, g) \in \mathbb{C}^B \oplus \mathbb{C}^W \mapsto h \in \mathbb{C}^E$ given by $h(e) = x(e)(f(b) - g(w))$, $e = bw \in E$. Consider cellular cochain complex

$$[\mathbb{C}^B \oplus \mathbb{C}^W \xrightarrow{\delta} \mathbb{C}^E]$$

with differential $\delta(f, g)(e) = g(w) - f(b)$, $f \in \mathbb{C}^B$, $g \in \mathbb{C}^W$, $e = bw \in E$, whose first cohomology is $H^1(\Gamma, \mathbb{C}) \cong \text{Loc}(\Gamma, \mathbb{C})$. Then we have an isomorphism of complexes

$$\begin{array}{ccc} \mathbb{C}^B \oplus \mathbb{C}^W & \longrightarrow & \mathbb{C}^E \\ \downarrow -\text{id} & & \downarrow \frac{1}{x} \cdot \\ \mathbb{C}^B \oplus \mathbb{C}^W & \longrightarrow & \mathbb{C}^E \end{array}$$

The following Theorem says that under this isomorphism, the differential $d\kappa$ is the identity map.

Theorem 8.4. *Let $\mathcal{L} = \kappa(x)$. The differential*

$$d\kappa : T_{[x]}\mathcal{X} \cong \text{Loc}(\Gamma, \mathbb{C}) \rightarrow T_{(C, S, \nu)}\mathcal{S} \cong \text{Ext}^1(\mathcal{L}, \mathcal{L})^G$$

is induced by the map $\dot{x} \mapsto x\dot{x}$, where $x\dot{x} \in \mathbb{C}^E$ (cf. Corollary 8.3).

Proof. We first describe explicitly how cocycles in $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$ parameterize deformations of \mathcal{L} , following [FIM12]. Take a local deformation of K^\bullet over U_i :

$$\left[\mathcal{E}|_{U_i} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \xrightarrow{\tilde{K}(x)+\varepsilon l_i} \mathcal{F}|_{U_i} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \right].$$

To get a global deformation of K^\bullet , we need to glue the local deformations along double intersections. This means we have isomorphisms $1 + m_{ij}\varepsilon$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}|_{U_{ij}} & \xrightarrow{\tilde{K}(x)+\varepsilon l_j} & \mathcal{F}|_{U_{ij}} \\ \downarrow 1+m_{ij}\varepsilon & & \downarrow 1+m_{ij}\varepsilon \cdot \\ \mathcal{E}|_{U_{ij}} & \xrightarrow{\tilde{K}(x)+\varepsilon l_i} & \mathcal{F}|_{U_{ij}} \end{array}$$

This gives the condition $l_j - l_i - (\tilde{K}(x)m_{ij} - m_{ij}\tilde{K}(x)) = 0$. Finally these isomorphisms have to satisfy the cocycle condition up to homotopy:

$$m_{ij} - m_{ik} + m_{jk} + (n_{ijk}\tilde{K}(x), \tilde{K}(x)n_{ijk}) = 0.$$

Now given a \mathbb{C} -local system $\frac{\dot{x}}{x}$, the deformation $\mathcal{L}' = d\kappa(x + \varepsilon\dot{x})$ of the sheaf \mathcal{L} is the cokernel

$$0 \rightarrow \mathcal{E} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \xrightarrow{\tilde{K}(x+\varepsilon\dot{x})} \mathcal{F} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \rightarrow \mathcal{L}' \rightarrow 0.$$

Notice that $\tilde{K}(x+\varepsilon\dot{x}) = \tilde{K}(x) + \varepsilon\tilde{K}(\dot{x})$. Therefore the cocycle representing \mathcal{L}' in $\text{Ext}^1(\mathcal{L}, \mathcal{L})^G$ is $(\tilde{K}(\dot{x}), 0, 0)$. \square

9 Computation of $\text{Ext}^\bullet(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$

The complex $[\mathcal{E} \xrightarrow{\tilde{K}(x)} \mathcal{F} \rightarrow \mathcal{L}|_D]$ is quasi-isomorphic to the complex $[\underline{\mathcal{L}} \rightarrow \mathcal{L}|_D]$. The complex $\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)$ is

$$\left[\mathcal{H}om(\mathcal{F}, \mathcal{E}) \xrightarrow{d^{-1}} \underline{\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F})} \xrightarrow{d^0} \mathcal{H}om(\mathcal{E}, \mathcal{F}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{L}|_D) \xrightarrow{d^1} \mathcal{H}om(\mathcal{E}, \mathcal{L}|_D) \right],$$

where the differentials are

$$d^{-1}(f) = (f\tilde{K}(x), \tilde{K}(x)f), \quad d^0(f, g) = (\tilde{K}(x)f - g\tilde{K}(x), rg), \quad d^1(f, g) = rf + g\tilde{K}(x), \quad (19)$$

where $r : \mathcal{L} \rightarrow \mathcal{L}|_D$ denotes the restriction map. The local-to-global Ext spectral sequence gives

$$\text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong \mathbb{H}^1(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G).$$

The double complex $\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G$ is

$$\begin{array}{ccccc} \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{L}|_D))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{L}|_D))^G & \longrightarrow & \\ \uparrow & & \uparrow & & \\ \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{L}|_D))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{L}|_D))^G & \longrightarrow & \\ \uparrow & & \uparrow & & \\ \underline{\check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G} & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}))^G & \longrightarrow & \\ \uparrow & & \uparrow & & \\ \check{C}^0(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{F}, \mathcal{E}))^G & \longrightarrow & \end{array} \quad (20)$$

A computation very similar to the proof of Proposition 8.2 gives:

Proposition 9.1. *The complex of vector spaces $R\text{Hom}(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ is quasi-isomorphic to*

$$F^\bullet := \left[\text{Hom}(\mathcal{F}, \mathcal{E})^G \rightarrow \underline{\text{Hom}(\mathcal{E}, \mathcal{E})^G \oplus \text{Hom}(\mathcal{F}, \mathcal{F})^G} \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F})^G \oplus \text{Hom}(\mathcal{F}, \mathcal{L}|_D)^G \rightarrow \text{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \right].$$

In particular, $\text{Ext}^i(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G = \mathcal{H}^i(F^\bullet)$.

Using Corollary 4.3 and the maps in Section 5.1, we get:

Corollary 9.2. *A representative for an element of $\text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ is given by a pair*

$$(\dot{x}, \phi_\alpha(w)) \in \mathbb{C}^E \oplus \bigoplus_{\alpha \in Z} \text{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C}) \cong \text{Hom}(\mathcal{E}, \mathcal{F})^G \oplus \text{Hom}(\mathcal{F}, \mathcal{L}|_D)^G$$

satisfying the following cocycle condition: For every zig-zag path α and wedge w_0, b, w_1 in α (see Figure 3), we have

$$\frac{\dot{b}}{b} - \frac{\dot{a}}{a} + \phi_\alpha(w_1) - \phi_\alpha(w_0) = 0.$$

Corollary 9.3. *A representative for an element of $\text{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ is given by an element of $\text{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C}) \cong \text{Hom}(\mathcal{E}, \mathcal{L}|_D)$.*

Corollary 9.2 may be interpreted as saying that $(\frac{\dot{x}}{x}, \phi_\alpha(w))$ defines a framed \mathbb{C} -local system (compare with (2)). This motivates the definition of the map

$$\lambda : \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \rightarrow \text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G; \quad (\dot{x}, \phi) \mapsto (x\dot{x}, \phi_\alpha(w)), \quad (21)$$

and it is straightforward to check that it is well-defined, i.e. that trivial \mathbb{C} -local systems map to coboundaries.

The following commuting diagram of complexes

$$\begin{array}{ccccc} \mathcal{L}(-D) & \xrightarrow{\theta} & \mathcal{L} & \xrightarrow{\text{id}} & \mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{L}|_D & \longrightarrow & 0 \end{array}$$

shows that the composition

$$\text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong \text{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \xrightarrow{\theta} \text{Ext}^1(\mathcal{L}, \mathcal{L})^G$$

is induced by the map of complexes

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\text{id}} & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{L}|_D & \longrightarrow & 0 \end{array}.$$

Therefore we have:

Proposition 9.4. *In terms of the representatives in Corollaries 8.3 and 9.2, the composition of maps*

$$\text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong \text{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \xrightarrow{\theta} \text{Ext}^1(\mathcal{L}, \mathcal{L})^G$$

is induced by the map

$$\begin{aligned} \mathbb{C}^E \oplus \bigoplus_{\alpha \in Z} \text{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C}) &\rightarrow \mathbb{C}^E \\ (\dot{x}, \phi_\alpha(w)) &\mapsto \dot{x}. \end{aligned}$$

Combining (1), (21), Theorem 8.4 and Proposition 9.4, we obtain:

Theorem 9.5. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Loc}^{\mathrm{fr}}(\Gamma, \mathbb{C}) & \xrightarrow{\mathrm{pr}_1} & \mathrm{Loc}(\Gamma, \mathbb{C}) \\
\downarrow \lambda & & \downarrow d\kappa \\
\mathrm{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G & \xrightarrow{\theta} & \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G
\end{array}$$

10 The Grothendieck-Serre duality pairing

The goal of this section is to compute the Grothendieck-Serre duality pairing

$$\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \xrightarrow{\circ} \mathrm{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \xrightarrow{\mathrm{tr}} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \xrightarrow{\cong} \mathbb{C}$$

using combinatorial data. We first state the result.

Proposition 10.1. *In terms of the representatives in Corollaries 8.3 and 9.2, the Grothendieck-Serre duality pairing $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \rightarrow \mathbb{C}$ is induced by the map*

$$\mathbb{C}^E \otimes (\mathbb{C}^E \oplus \mathrm{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C})) \rightarrow \mathbb{C}$$

$$\dot{x} \otimes (\dot{y}, \phi_\alpha(w)) \mapsto \sum_{\alpha \in Z} \sum_{b \in B \cap \alpha} \left(\frac{\dot{x}(bw_0)}{x(bw_0)} \phi_\alpha(w_0) - \frac{\dot{x}(bw_1)}{x(bw_1)} \phi_\alpha(w_1) \right), \quad (22)$$

where w_0, b, w_1 is a wedge in α as in Figure 3.

The rest of this section is devoted to the proof of this proposition. The proof is a long sequence of homological algebra computations, and the reader might prefer to skip it on first reading.

Our goals are two-fold: we need to be able to perform the homological algebra computations, and also be able to interpret the results combinatorially. We motivate the argument now. If we choose a Kasteleyn matrix $\tilde{K}(x)$ and the isomorphism of complexes $[\mathcal{L}(-D)] \cong [\mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{L}|_D]$, we can compute the vector spaces $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G$, $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G$ and $\mathrm{Ext}^2(\mathcal{L}, \mathcal{L}(-D))^G$ as in Corollaries 8.3, 9.2 and 9.3 as the hypercohomology groups $\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet))^G)$ and $\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G)$, which we understand combinatorially as the spaces of local systems and framed local systems. Moreover, as we saw in Propositions 8.2 and 9.1, the cohomology of both these double complexes is only in the degree 0, which makes it especially easy to compute the cup product. As we show in the Appendix, the space $\mathrm{Loc}^{\mathrm{fr}}(\Gamma, \mathbb{C})$ is isomorphic to $H_1(\Gamma, \mathbb{C})$, and as we show in Section 10.3, if we compute $\mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L}(-D))^G$ as the hypercohomology group

$\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$, we get a description that can be understood combinatorially as $H_1(\Gamma, \mathbb{C})$. However the cohomology of $\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$ is only in degree 2 (Proposition 10.3), and this makes the computation of the cup product in terms of the Čech double complex very complicated. This observation is the reason for using the framed local systems description rather than the homology description of the cotangent space although Poincaré duality is more directly understood in terms of the latter description.

Therefore in subsection 10.1, we use the local systems descriptions of the domain and codomain spaces to compute the cup product

$$\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \rightarrow \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}(-D))^G.$$

In Subsection 10.2, we compute the trace map which is defined naturally in terms of the hypercohomology group $\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$. Therefore, to compose the cup product and the trace, we need to compute the natural isomorphism $\Psi : \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)))^G \cong \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$, which is an algebro-geometric avatar of the isomorphism between framed local systems and homology. We do this in a round-about way: in Subsection 10.3, we define a map $\tau : \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)))^G \rightarrow \mathbb{C}$, which is to be the composition $\mathrm{tr} \circ \Psi$ and whose form we can guess (see Notation 10.5) since combinatorially, upon composing with the cup product, it must give the Poincaré duality pairing. Finally, we observe that both the \mathbb{H}^2 -spaces are one dimensional, so we can show that τ and $\mathrm{tr} \circ \Psi$ agree by checking it for a single element, which we do in Section 10.3.

10.1 The cup product

The main result for this subsection is the following:

Lemma 10.2. *In terms of the representatives in Corollaries 8.3, 9.2 and 9.3, the cup product map $\circ : \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \rightarrow \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}(-D))^G$ is induced by the map*

$$\begin{aligned} \mathbb{C}^E \otimes (\mathbb{C}^E \oplus \mathrm{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C})) &\rightarrow \mathrm{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C}) \\ \dot{x} \otimes (\dot{y}, \phi_\alpha(\mathbf{w})) &\mapsto \eta_\alpha(\mathbf{b}), \end{aligned}$$

where $(\eta_\alpha(\mathbf{b}_1), \dots, \eta_\alpha(\mathbf{b}_d)) : \mathbb{C}^{B \cap \alpha} \rightarrow \mathbb{C}$ is given by

$$\left(\frac{\dot{a}_1}{a_1} \phi_\alpha(\mathbf{w}_1) - \frac{\dot{b}_1}{b_1} \phi_\alpha(\mathbf{w}_2), \dots, \frac{\dot{a}_d}{a_d} \phi_\alpha(\mathbf{w}_d) - \frac{\dot{b}_d}{b_d} \phi_\alpha(\mathbf{w}_1) \right),$$

where \dot{x}, x on the edges of α are denoted as in Figure 7.

Proof. From [HL10, Section 10.1.1], the map $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \rightarrow \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}(-D))^G$ is obtained using the composition

$$R\mathcal{H}om(\mathcal{L}, \mathcal{L})^G \otimes^L R\mathcal{H}om(\mathcal{L}, \mathcal{L}(-D))^G \rightarrow R\mathcal{H}om(\mathcal{L}, \mathcal{L}(-D))^G$$

together with

$$H^\bullet(R\mathcal{H}om(\mathcal{L}, \mathcal{L})^G) \otimes H^\bullet(R\mathcal{H}om(\mathcal{L}, \mathcal{L}(-D))^G) \rightarrow H^\bullet(R\mathcal{H}om(\mathcal{L}, \mathcal{L})^G \otimes^L R\mathcal{H}om(\mathcal{L}, \mathcal{L}(-D))^G).$$

The first map is the usual composition of functions, whereas the second one can be described at the level of Čech cocycles, using (16) and the isomorphism $\mathcal{L}(-D) \cong [\mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{L}|_D]$.

Indeed, each row of (16) has no cohomology in degree different from 0, so it suffices to describe the desired map on the cohomology of the first column. This is the map

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}, \mathcal{F})^G \otimes \left(\mathrm{Hom}(\mathcal{E}, \mathcal{F})^G \oplus \mathrm{Hom}(\mathcal{F}, \mathcal{L}|_D)^G \right) &\rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{L}|_D)^G \\ \tilde{K}(\dot{x}) \otimes \left(\tilde{K}(\dot{y}), f \right) &\mapsto f \circ \tilde{K}(\dot{x}). \end{aligned}$$

Now we use the identifications in Corollaries 8.3, 9.2 and 9.3. Suppose $f \mapsto \phi_\alpha(\mathbf{w}) \in \mathrm{Hom}(\mathbb{C}^{W \cap \alpha}, \mathbb{C})$. Then the composition $f \circ \tilde{K}(\dot{x})$ becomes

$$[\phi_\alpha(\mathbf{w}_1) \quad \cdots \quad \phi_\alpha(\mathbf{w}_d)] \begin{bmatrix} \frac{\dot{a}_1}{a_1} & & & -\frac{\dot{b}_d}{b_d} \\ -\frac{\dot{b}_1}{b_1} & \frac{\dot{a}_2}{a_2} & & \\ & -\frac{\dot{b}_2}{b_2} & \frac{\dot{a}_3}{a_3} & \\ & & \ddots & \ddots \end{bmatrix} = \left[\frac{\dot{a}_1}{a_1} \phi_\alpha(\mathbf{w}_1) - \frac{\dot{b}_1}{b_1} \phi_\alpha(\mathbf{w}_2) \quad \cdots \quad \frac{\dot{a}_d}{a_d} \phi_\alpha(\mathbf{w}_d) - \frac{\dot{b}_d}{b_d} \phi_\alpha(\mathbf{w}_1) \right].$$

□

10.2 The trace map

To compute the trace map, we use the locally free resolution $K^\bullet(-D) \xrightarrow{\theta} [\mathcal{L} \rightarrow \mathcal{L}|_D]$. Therefore the first step is to compute the complex $R\mathrm{Hom}(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ in terms of this locally free resolution.

The complex $\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))$ is

$$\left[\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)) \xrightarrow{d^{-1}} \underline{\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D))} \xrightarrow{d^0} \mathcal{H}om(\mathcal{E}, \mathcal{F}(-D)) \right],$$

where the differentials are $d^{-1}(f) = (f\tilde{K}(x), \tilde{K}(x)f)$ and $d^0(f, g) = \tilde{K}(x)f - g\tilde{K}(x)$.

Proposition 10.3. *The complex of vector spaces $R\mathrm{Hom}(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ is quasi-isomorphic to*

$$\left[\underline{0} \rightarrow H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D))) \rightarrow H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D))) \right].$$

In particular, $\mathrm{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G / H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G$.

Proof. Using the local to global Ext spectral sequence, $R\mathrm{Hom}(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G$ is the hypercohomology $\mathbb{H}^\bullet(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G)$, which we compute using spectral sequences. The E_0 -page is

$$\begin{array}{ccccc}
\check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}(-D)))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}(-D)))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{F}(-D)))^G \\
\uparrow & & \uparrow & & \uparrow \\
\check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G \\
\uparrow & & \uparrow & & \uparrow \\
\check{C}^0(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G & \longrightarrow & \check{C}^1(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G & \longrightarrow & \check{C}^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G
\end{array} \tag{23}$$

Taking G -invariant Čech cohomology and noticing that all the but two of the cohomology groups vanish (either directly computing them, or using that the cohomology groups are Serre dual to the ones in 18), we get the E_1 -page

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G
\end{array} \tag{24}$$

□

We have by Example 3.19,

$$H^2(\mathcal{H}om(\mathcal{E}_b, \mathcal{E}_{b'}(-D)))^G \cong \begin{cases} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \cong \mathbb{C} & \text{if } b = b', \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned}
H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)))^G &\cong \bigoplus_{b \in B} H^2(\mathcal{H}om(\mathcal{E}_b, \mathcal{E}_b(-D)))^G \\
&\cong \bigoplus_{b \in B} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G,
\end{aligned}$$

and similarly

$$\begin{aligned}
H^2(\mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G &\cong \bigoplus_{w \in W} H^2(\mathcal{H}om(\mathcal{F}_w, \mathcal{F}_w(-D)))^G \\
&\cong \bigoplus_{w \in W} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G.
\end{aligned}$$

Therefore a 2-Čech-cocycle representing a cohomology class in $H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G$ is of the form

$$\left(\frac{f(\mathbf{b})}{x_0 x_1 x_2}, \frac{g(\mathbf{w})}{x_0 x_1 x_2} \right)_{\mathbf{b} \in B, \mathbf{w} \in W},$$

for $f \in \mathbb{C}^B, g \in \mathbb{C}^W$.

Lemma 10.4. *Under the isomorphism $\text{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \cong \mathbb{H}^2(\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G)$, the trace map $\text{tr} : \text{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G$ is induced by*

$$\begin{aligned} H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G &\rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \\ \left(\frac{f(\mathbf{b})}{x_0 x_1 x_2}, \frac{g(\mathbf{w})}{x_0 x_1 x_2} \right) &\mapsto \frac{1}{x_0 x_1 x_2} \left(\sum_{\mathbf{w} \in W} g(\mathbf{w}) - \sum_{\mathbf{b} \in B} f(\mathbf{b}) \right). \end{aligned}$$

Proof. Consider the trace map (cf. [HL10, Section 10.1.2]) $\text{tr} : \mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))^G \rightarrow \left[\mathcal{O}_{\mathbb{P}^2}^G(-D) \right]$, where $\left[\mathcal{O}_{\mathbb{P}^2}^G(-D) \right]$ denotes the complex with $\mathcal{O}_{\mathbb{P}^2}^G(-D)$ in degree 0, defined as

$$\text{tr}|_{\mathcal{H}om(K^i, K^j(-D))} = \begin{cases} (-1)^i \text{tr}_{K^i} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{tr}_{K^i} : \mathcal{H}om(K^i, K^i(-D)) \rightarrow \mathcal{O}_{\mathbb{P}^2}^G(-D)$ is the usual trace map. The induced map on cohomology

$$\text{tr} : \text{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G$$

can be computed using map of Čech double complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{O}_{\mathbb{P}^2}^G(-D)).$$

Taking G -invariant Čech cohomology, we see that the E_1 -pages of both sides are nonzero only in the second column (see (24)), and the induced map of E_1 -pages is the one in the statement of the Lemma. \square

10.3 Tic-tac-toe

The codomain of the cup product as described in Lemma 10.2 is $\mathbb{H}^2(\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G)$ whereas the domain of the trace map in Lemma 10.4 is $\mathbb{H}^2(\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G)$. Therefore we need to compute the composition

$$\Phi : \mathbb{H}^2(\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G) \cong \mathbb{H}^2(\check{\mathcal{C}}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G) \xrightarrow{\text{tr}} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \xrightarrow{\cong} \mathbb{C}. \quad (25)$$

We have $H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G \cong \mathbb{C}^E$, and $H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G \cong \mathbb{C}^B \oplus \mathbb{C}^W$, and the complex

$$[H^2(\mathcal{H}om(\mathcal{F}, \mathcal{E}(-D)))^G \rightarrow H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G]$$

is isomorphic to the cellular chain complex $[\mathbb{C}^E \xrightarrow{\partial} \mathbb{C}^B \oplus \mathbb{C}^W]$ used to compute the homology of Γ , so we have

$$\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G \cong H_0(\Gamma, \mathbb{C}) \cong \mathbb{C}.$$

Therefore all the maps in (25) are isomorphisms as they are non-zero.

Notation 10.5. We define $\tau : \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)))^G \rightarrow \mathbb{C}$ be the map given by

$$\text{Hom}(\mathbb{C}^{B \cap \alpha}, \mathbb{C}) \rightarrow \mathbb{C}; \quad \eta_\alpha(b) \mapsto \sum_{\alpha \in Z} \sum_{b \in B \cap \alpha} \eta_\alpha(b)$$

Observe that τ is nonzero, and therefore an isomorphism. Since the maps τ and Φ are both isomorphisms between one dimensional complex vector spaces, they are the same modulo a multiplicative constant.

We need to work much harder to find the multiplicative constant, since it could in principle depend on \mathcal{L} , and we would only get the weaker statement that the Poisson bracket is given by (regular function) $\cdot \theta$. However, we will see that the multiplicative constant is 1.

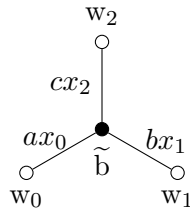
To find the multiplicative constant relating τ and Φ , we compare the images of a certain simple element of $\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$ under the two maps

$$\begin{aligned} \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G &\xrightarrow{\text{tr}} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \xrightarrow{\cong} \mathbb{C}, \\ \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G &\xrightarrow{\cong} \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)))^G \xrightarrow{\tau} \mathbb{C}. \end{aligned}$$

Let

$$\left[\left(\frac{\delta_{\tilde{b}b} \delta_{\tilde{b}b'}}{x_0 x_1 x_2}, 0 \right) \right] \in H^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)) \oplus \mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G. \quad (26)$$

Recall that in homogeneous coordinates, $\theta = x_0 x_1 x_2$ and the Kasteleyn matrix \tilde{K} near \tilde{b} takes the form



where $a := x(\tilde{b}w_0)$, $b := x(\tilde{b}w_1)$, $c := x(\tilde{b}w_2)$.

Lemma 10.6. *Under the composition*

$$\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G \xrightarrow{\text{tr}} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \xrightarrow[\cong]{\theta} \mathbb{C},$$

the element (26) maps to -1 .

Proof. Direct computation using Lemma 10.4. □

Lemma 10.7. *Under the composition*

$$\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G \xrightarrow{\cong} \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D)))^G \xrightarrow{\tau} \mathbb{C},$$

the element (26) maps to -1 .

Proof. This involves two applications of the “tic-tac-toe lemma”, as explained in [BT82, Proof of Proposition 12.1].

A 2-cocycle for the hypercohomology $\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D))))^G$ representing $\left[\left(\frac{\delta_{\tilde{b}b} \delta_{\tilde{b}b'}}{x_0 x_1 x_2}, 0 \right) \right]$ is given by (cf. (23) for the double complex)

1		$\zeta(\mathbf{b}, \mathbf{w})_{ij}$	
0			$(\xi(\mathbf{b}, \mathbf{b}')_{012}, \xi(\mathbf{w}, \mathbf{w}')_{012})$
-1			
	0	1	2

where $\zeta(\mathbf{b}, \mathbf{w})_{ij} \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}(-D)))^G$ is given by

$$(\zeta(\mathbf{b}, \mathbf{w})_{01}, \zeta(\mathbf{b}, \mathbf{w})_{02}, \zeta(\mathbf{b}, \mathbf{w})_{12}) = \begin{cases} \left(0, 0, -\frac{a}{x_1 x_2} \right) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_0, \\ \left(0, \frac{b}{x_0 x_2}, 0 \right) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_1, \\ \left(-\frac{c}{x_0 x_1}, 0, 0 \right) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_2, \\ (0, 0, 0) & \text{otherwise,} \end{cases}$$

and

$$\xi(\mathbf{b}, \mathbf{b}')_{012} = \frac{\delta_{\tilde{b}b} \delta_{\tilde{b}b'}}{x_0 x_1 x_2} \in \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}(-D)))^G; \quad \xi(\mathbf{w}, \mathbf{w}')_{012} = 0 \in \check{C}^2(\mathcal{H}om(\mathcal{F}, \mathcal{F}(-D)))^G,$$

where the $\zeta(\mathbf{b}, \mathbf{w})_{ij}$ are chosen so that the hypercohomology cocycle conditions

$$\begin{aligned}\zeta(\tilde{\mathbf{b}}, \mathbf{w}_1)_{01} - \zeta(\tilde{\mathbf{b}}, \mathbf{w}_1)_{02} + \zeta(\tilde{\mathbf{b}}, \mathbf{w}_1)_{12} + \frac{ax_0}{x_0x_1x_2} &= 0, \\ \zeta(\tilde{\mathbf{b}}, \mathbf{w}_2)_{01} - \zeta(\tilde{\mathbf{b}}, \mathbf{w}_2)_{02} + \zeta(\tilde{\mathbf{b}}, \mathbf{w}_2)_{12} + \frac{bx_1}{x_0x_1x_2} &= 0, \\ \zeta(\tilde{\mathbf{b}}, \mathbf{w}_3)_{01} - \zeta(\tilde{\mathbf{b}}, \mathbf{w}_3)_{02} + \zeta(\tilde{\mathbf{b}}, \mathbf{w}_3)_{12} + \frac{cx_2}{x_0x_1x_2} &= 0,\end{aligned}$$

are satisfied.

A 2-cocycle for the image of $\left[\left(\frac{\delta_{\tilde{\mathbf{b}}\tilde{\mathbf{b}}'}}{x_0x_1x_2}, 0\right)\right]$ under the map

$$\mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet(-D)))^G) \xrightarrow[\cong]{\theta} \mathbb{H}^2(\check{C}^\bullet(\mathcal{H}om^\bullet(K^\bullet, K^\bullet \rightarrow \mathcal{L}|_D))^G)$$

is obtained by multiplying every term in the double complex by $\theta = x_0x_1x_2$, and is given by (cf. (20) for the double complex)

2	$(\eta_{\tilde{\alpha}_l}(\mathbf{b})_i, \mu_{\tilde{\alpha}_l}(\mathbf{w})_i)$		
1		$\zeta'(\mathbf{b}, \mathbf{w})_{ij}$	
0			$(\xi'(\mathbf{b}, \mathbf{b}')_{012}, \xi'(\mathbf{w}, \mathbf{w}')_{012})$
-1			
	0	1	2

where

$$\begin{aligned}\eta_{\tilde{\alpha}_l}(\mathbf{b}) &= 0 \in \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{L}|_D))^G, \\ \mu_{\tilde{\alpha}_l}(\mathbf{w}) &= 0 \in \check{C}^0(\mathcal{H}om(\mathcal{F}, \mathcal{L}|_D))^G, \\ \zeta'(\mathbf{b}, \mathbf{w}) &= x_0x_1x_2\zeta(\mathbf{b}, \mathbf{w}) \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G, \\ \xi'(\mathbf{b}, \mathbf{b}') &= \delta_{\tilde{\mathbf{b}}\tilde{\mathbf{b}}'} \in \check{C}^2(\mathcal{H}om(\mathcal{E}, \mathcal{E}))^G, \\ \xi'(\mathbf{w}, \mathbf{w}') &= 0 \in \check{C}^2(\mathcal{H}om(\mathcal{F}, \mathcal{F}))^G.\end{aligned}$$

Explicitly, $\zeta'(\mathbf{b}, \mathbf{w})_{ij}$ is given by

$$(\zeta'(\mathbf{b}, \mathbf{w})_{01}, \zeta'(\mathbf{b}, \mathbf{w})_{02}, \zeta'(\mathbf{b}, \mathbf{w})_{12}) = \begin{cases} (0, 0, -ax_0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_0, \\ (0, bx_1, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_1, \\ (-cx_2, 0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_2, \\ (0, 0, 0) & \text{otherwise.} \end{cases}$$

Finally we modify this 2-cocycle by a 2-coboundary so that only the 0th column has nonzero components. Let $\delta = \delta_{\rightarrow} + \delta_{\uparrow}$ denote the differential of the total complex of the double complex, where $\delta_{\rightarrow} = \check{d}$ and $\delta_{\uparrow} = (-1)^p d$, where d is given in (19). Generally a 2-coboundary is the image under δ of a 5-tuple, but we only have two nonzero components below. Consider $(\delta_{\tilde{b}\tilde{b}}\tilde{\delta}_{\tilde{b}\tilde{b}'}, 0, 0) \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{E}))$, whose image under δ is given by

2	$(0, 0)$		
1		$\zeta'_{(1)}(\mathbf{b}, \mathbf{w})_{ij}$	
0		$(-\delta_{\tilde{b}\tilde{b}}, 0, 0)$	$(-\delta_{\tilde{b}\tilde{b}}\tilde{\delta}_{\tilde{b}\tilde{b}'}, 0)$
-1			
	0	1	2

where $\zeta'_{(1)}(\mathbf{b}, \mathbf{w})_{ij} \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G$ is given by

$$(\zeta'_{(1)}(\mathbf{b}, \mathbf{w})_{01}, \zeta'_{(1)}(\mathbf{b}, \mathbf{w})_{02}, \zeta'_{(1)}(\mathbf{b}, \mathbf{w})_{12}) = \begin{cases} (ax_0, 0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_0, \\ (bx_1, 0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_1, \\ (cx_2, 0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_2, \\ (0, 0, 0) & \text{otherwise,} \end{cases}$$

Next, we take $\omega(\mathbf{b}, \mathbf{w})_i \in \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$, where

$$(\omega(\mathbf{b}, \mathbf{w})_0, \omega(\mathbf{b}, \mathbf{w})_1, \omega(\mathbf{b}, \mathbf{w})_2) = \begin{cases} (0, -ax_0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_0, \\ (bx_1, 0, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_1, \\ (0, 0, 0) & \text{otherwise,} \end{cases}$$

whose image under δ is given by

2	$(\eta'_{\tilde{\alpha}_i}(\mathbf{b})_i, 0)$		
1	$\omega(\mathbf{b}, \mathbf{w})_i$	$\zeta'_{(2)}(\mathbf{b}, \mathbf{w})_{ij}$	
0			$(0, 0)$
-1			
	0	1	2

where $\zeta'_{(2)}(\mathbf{b}, \mathbf{w})_{ij} \in \check{C}^1(\mathcal{H}om(\mathcal{E}, \mathcal{F}))^G$ is given by

$$\zeta'_{(2)}(\mathbf{b}, \mathbf{w})_{01}, \zeta'_{(2)}(\mathbf{b}, \mathbf{w})_{02}, \zeta'_{(2)}(\mathbf{b}, \mathbf{w})_{12} = \begin{cases} (-ax_0, 0, ax_0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_0, \\ (-bx_1, -bx_1, 0) & \text{if } \mathbf{b} = \tilde{\mathbf{b}} \text{ and } \mathbf{w} = \mathbf{w}_1, \\ (0, 0, 0) & \text{otherwise,} \end{cases}$$

and $\eta'_{\tilde{\alpha}_l}(\mathbf{b})_i \in \check{C}^0(\mathcal{H}om(\mathcal{E}, \mathcal{L}|_D))^G$ is given by $(\eta'_{\tilde{\alpha}_l}(\mathbf{b})_0, \eta'_{\tilde{\alpha}_l}(\mathbf{b})_1, \eta'_{\tilde{\alpha}_l}(\mathbf{b})_2)$ equal to

$$\left(\mathcal{E}_{\mathbf{b}}|_{U_0} \xrightarrow{bx_1} \mathcal{F}_{\mathbf{w}_1}|_{U_0} \xrightarrow{\psi_{\tilde{\alpha}_l}(\mathbf{w}_1)} \mathcal{L}|_{\tilde{\alpha}_l}, \mathcal{E}_{\mathbf{b}}|_{U_1} \xrightarrow{-ax_0} \mathcal{F}_{\mathbf{w}_0}|_{U_1} \xrightarrow{\psi_{\tilde{\alpha}_l}(\mathbf{w}_0)} \mathcal{L}|_{\tilde{\alpha}_l}, 0 \right) \quad (27)$$

if α is the zig-zag path through $\mathbf{w}_0 \tilde{\mathbf{b}} \mathbf{w}_1$ and $\mathbf{b} = \tilde{\mathbf{b}}$ and $(0, 0, 0)$ otherwise, where $\psi_{\tilde{\alpha}_l}(\mathbf{w})$ denotes the map $\mathcal{F}_{\mathbf{w}} \rightarrow \mathcal{L}|_{\tilde{\alpha}_l}$.

Adding both the 2-coboundaries to the 2-cocycle, we get the 2-cocycle $(\eta'_{\tilde{\alpha}_l}(\mathbf{b})_i, 0, 0, 0, 0)$. Now notice that the first two maps in (27) are equal, since \mathcal{L} is the cokernel of the Kasteleyn matrix, and that $\tilde{\alpha}_l \notin U_2$. Therefore $d(\eta'_{\tilde{\alpha}_l}(\mathbf{b})_i) = 0$, and so $\eta'_{\tilde{\alpha}_l}(\mathbf{b})_i$ gives a global section of $\mathcal{H}om(\mathcal{E}, \mathcal{L}|_D)$, which under the isomorphism of Lemma 5.1 is $\mathbb{C}_{\mathbf{b}} \xrightarrow{-1} \mathbb{C}$. Therefore under τ , we get -1 . \square

Proof of Proposition 10.1. From Lemmas 10.6 and 10.7, we get that the map

$$\text{Ext}^2(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G \xrightarrow{\text{tr}} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^G(-D))^G \xrightarrow{\cong} \mathbb{C}.$$

is equal to τ . Composing the cup product map in Lemma 10.2 with τ gives (22). \square

11 Conclusion of the proof

From Proposition 2.3, Remark 2.4 and Proposition 10.1, we see that the diagram

$$\begin{array}{ccc} \text{Loc}(\Gamma, \mathbb{C}) \otimes \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) & \xrightarrow{\text{Poincaré duality}} & \mathbb{C} \\ \downarrow d\kappa \otimes \lambda & & \parallel \\ \text{Ext}^1(\mathcal{L}, \mathcal{L})^G \otimes \text{Ext}^1(\mathcal{L}, \mathcal{L} \rightarrow \mathcal{L}|_D)^G & \xrightarrow{\text{Grothendieck-Serre duality}} & \mathbb{C} \end{array}$$

commutes, which implies that the diagram

$$\begin{array}{ccc} \text{Loc}(\Gamma, \mathbb{C})^* & \xrightarrow[\cong]{\text{Poincaré duality}} & \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \\ \uparrow d\kappa^* & & \downarrow \lambda \\ (\text{Ext}^1(\mathcal{L}, \mathcal{L})^G)^* & \xrightarrow[\cong]{\text{Grothendieck-Serre duality}} & \text{Ext}^1(\mathcal{L}, \mathcal{L}(-D))^G \end{array}$$

commutes. Combined with Theorem 9.5, we get commutativity of (14), and the proof of Theorem 7.1 is complete.

Appendix A Discrete local systems vs cellular cohomology

The goal of this Appendix is to translate the description of the Poisson structure in [GK13] in terms of the intersection form on the conjugate surface to the setting of local systems and framed local systems. This is possibly well known.

Let \mathbb{C}^Z denote the space of functions on zig-zag paths. Given a function $f \in \mathbb{C}^Z$, we define a framed local system (y, ϕ) as follows:

1. Let $y(e) = 0$ for all $e \in E$.
2. Suppose α is a zig-zag path and $w \in W \cap \alpha$. Define $\phi_\alpha(w) = f(\alpha)$.

We have constructed a map $\mathbb{C}^Z \rightarrow \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C})$, whose kernel consists of constant functions. The following sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^Z \rightarrow \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \xrightarrow{\text{pr}_1} \text{Loc}(\Gamma, \mathbb{C}) \rightarrow \left\{ x \in \bigoplus_{\alpha \in Z} \text{Loc}(\alpha, \mathbb{C}) : \sum_{\alpha \in Z} [x](\alpha) = 0 \right\} \rightarrow 0, \quad (28)$$

is exact, where the leftmost map is inclusion of constant functions, and the rightmost map is restriction of local systems to zig-zag paths.

Consider a pair of topological spaces (X, A) where $A \subset X$. We have the short exact sequence of chains

$$0 \rightarrow C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{j_*} C_n(X, A) := C_n(X)/C_n(A) \rightarrow 0,$$

where i_* comes from the inclusion $i : A \hookrightarrow X$. Dualizing by applying $\text{Hom}(\cdot, \mathbb{C})$, we get a short exact sequence of cochains

$$0 \rightarrow C^n(X, A, \mathbb{C}) \xrightarrow{j^*} C^n(X, \mathbb{C}) \xrightarrow{i^*} C^n(A) \rightarrow 0,$$

which gives the long exact sequence of cohomology groups

$$\cdots \rightarrow H^n(X, A, \mathbb{C}) \xrightarrow{j^*} H^n(X, \mathbb{C}) \xrightarrow{i^*} H^n(A, \mathbb{C}) \xrightarrow{\delta} H^{n+1}(X, A, \mathbb{C}) \rightarrow \cdots$$

Let $X = \widehat{\Gamma}$ and $A = \partial\widehat{\Gamma}$. We have the following relevant portion of the long exact sequence

$$0 \rightarrow H^0(\widehat{\Gamma}, \mathbb{C}) \xrightarrow{i^*} H^0(\partial\widehat{\Gamma}, \mathbb{C}) \xrightarrow{\delta} H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}) \xrightarrow{j^*} H^1(\widehat{\Gamma}, \mathbb{C}) \xrightarrow{i^*} \text{Ker } \delta \rightarrow 0. \quad (29)$$

Let us describe the groups appearing in (29).

1. For a topological space X , $H^0(X, \mathbb{C})$ is the space of locally constant functions on X . Since $\widehat{\Gamma}$ is connected, $H^0(\widehat{\Gamma}, \mathbb{C}) \cong \mathbb{C}$ is the space of constant functions.
2. Since the components of $\widehat{\Gamma}$ are in canonical bijection with Z , $H^0(\partial\widehat{\Gamma}, \mathbb{C}) \cong \mathbb{C}^Z$. The map $i^* : H^0(\widehat{\Gamma}, \mathbb{C}) \rightarrow H^0(\partial\widehat{\Gamma}, \mathbb{C})$ is the inclusion of constant functions.

3. $\ker \delta$ denotes the kernel of $\delta : H^1(\partial\widehat{\Gamma}, \mathbb{C}) \rightarrow H^2(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C})$ and consists of cohomology classes on zig-zag paths whose product is 1.

We have the following version of Poincaré duality for manifolds with boundary.

Theorem A.1 ([Hat02, Theorem 3.43]). *Suppose M is a compact orientable n -manifold with boundary ∂M a compact $(n-1)$ -manifold. We have an isomorphism (the intersection pairing)*

$$H^k(M, \partial M, \mathbb{C}) \cong H_{n-k}(M, \mathbb{C}).$$

We now recall the definition of the conjugate ribbon graph $\widehat{\Gamma}$ from [GK13, Section 1.1.1]. Thickening the edges of Γ , we obtain a ribbon graph. The ribbon graph obtained from Γ has the cyclic order induced from the embedding in the torus \mathbb{T} . Let $\widehat{\Gamma}$ be the ribbon graph obtained from Γ by reversing the cyclic order at all white vertices. Alternately, to obtain $\widehat{\Gamma}$, we cut each edge of the ribbon graph and glue them back with a twist. The boundary components of $\widehat{\Gamma}$ are canonically in bijection with the zig-zag paths of Γ .

Letting $M = \widehat{\Gamma}$ in Theorem A.1, we get

$$H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}) \cong H_1(\widehat{\Gamma}, \mathbb{C}).$$

One can check that the following is a map of complexes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^Z & \longrightarrow & \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) & \xrightarrow{\text{pr}_1} & \text{Loc}(\Gamma, \mathbb{C}) & \longrightarrow & \{\sum_{\alpha \in Z} [x][\alpha] = 0\} & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & H^0(\widehat{\Gamma}, \mathbb{C}) & \longrightarrow & H^0(\partial\widehat{\Gamma}, \mathbb{C}) & \longrightarrow & H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}) & \longrightarrow & H^1(\widehat{\Gamma}, \mathbb{C}) & \longrightarrow & \text{Ker } \delta & \longrightarrow & 0
\end{array} \tag{30}$$

where all the down maps are the obvious ones, except for the middle down map which is the composition

$$\text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \rightarrow H_1(\Gamma, \mathbb{C}) \cong H_1(\widehat{\Gamma}, \mathbb{C}) \cong H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}), \tag{31}$$

where the map $\text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \rightarrow H_1(\Gamma, \mathbb{C})$ is given by $[(y, \phi)] \mapsto (a(e))_{e \in E} \in C_1(\Gamma, \mathbb{C})$, where $a(e) = \phi_{\alpha_l}(e) - \phi_{\alpha_r}(e)$ and α_l, α_r were defined in Section 2.3. Since for any vertex $v \in B \sqcup W$, we have $\sum_{v \sim e} a(e) = 0$, $(a(e))_{e \in E}$ is a cycle, and therefore gives a homology class.

By the five lemma, the map (31) is an isomorphism, and therefore the map $\text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \rightarrow H_1(\Gamma, \mathbb{C})$ is also an isomorphism. Moreover the following diagram is commutative

$$\begin{array}{ccc}
\text{Loc}(\Gamma, \mathbb{C}) \otimes \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) & \longrightarrow & \mathbb{C} \\
\downarrow \cong & & \parallel \\
H^1(\widehat{\Gamma}, \mathbb{C}) \otimes H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}) & \xrightarrow{\text{Poincaré duality}} & \mathbb{C}
\end{array} ,$$

where the pairing $\text{Loc}(\Gamma, \mathbb{C}) \otimes \text{Loc}^{\text{fr}}(\Gamma, \mathbb{C}) \rightarrow \mathbb{C}$ is the one in Proposition 2.3, since both pairings are equal, under the respective isomorphisms with $H^1(\Gamma, \mathbb{C}) \otimes H_1(\Gamma, \mathbb{C})$, to the pairing $(x(e))_{e \in E} \otimes (a(e))_{e \in E} \mapsto \sum_{e \in E} x(e)a(e)$. This justifies the name Poincaré duality for the pairing in Proposition 2.3.

The Poisson structure on \mathcal{X} from [GK13, Section 1.1.1] is defined as the composition

$$H^1(\Gamma, \mathbb{C})^* = H_1(\Gamma, \mathbb{C}) \cong H_1(\widehat{\Gamma}, \mathbb{C}) \cong H^1(\widehat{\Gamma}, \partial\widehat{\Gamma}, \mathbb{C}) \xrightarrow{j^*} H^1(\widehat{\Gamma}, \mathbb{C}) \cong H^1(\Gamma, \mathbb{C}),$$

which, under the isomorphisms in (30), coincides with (3).

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