Spectra of biperiodic planar networks

Terrence George

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Abstract

A biperiodic planar resistor network is a pair (G, c) where G is a graph embedded on the torus and c is a function from the edges of G to non-zero complex numbers. Associated with the discrete Laplacian on a biperiodic planar network is its spectral data: a triple (C, S, ν), where C is a curve, and S is a divisor on it, which we show is a point in the Prym variety of C. We give a complete classification of networks (modulo a natural equivalence) in terms of their spectral data. The space of networks has a large group of cluster automorphisms arising from the Y-Δ transformation, giving discrete cluster integrable systems. We show that these automorphisms are integrable in the algebro-geometric sense: under the spectral transform, they become translations in the Prym variety.

1 Introduction

A *planar resistor network* is a pair $(\tilde{G}, \tilde{c})$ where $\tilde{G}$ is a planar graph and $\tilde{c}$ is a conductance function that assigns a non-zero complex number to each edge of $\tilde{G}$, defined up to multiplication by a global constant. It is said to be *biperiodic* if translations by $\mathbb{Z}^2$ act on $(\tilde{G}, \tilde{c})$ by isomorphisms. This is equivalent to the data of the quotient $(G, c) := (\tilde{G}, \tilde{c})/\mathbb{Z}^2$, where G is a graph on a torus and c is a conductance function on G. Hereafter, we assume that our networks are on a torus.

The fundamental operator in the study of networks is the discrete Laplacian. It is a periodic finite-difference operator associated to which is its spectral data, a curve, and a divisor on it, defined below. The main goal of this paper is to show that the *spectral transform*, the map that takes a biperiodic network to its spectral data, is a birational map from the space of biperiodic networks to a certain moduli space of curves and divisors. Therefore, the spectral transform provides a classification of networks in the torus, analogous to the classification of resistor networks in the disk in terms of their response matrices due to De Verdière-Gitler-Vertigan [CdVGV96] and Curtis-Ingerman-Morrow [CIM98], and in a cylinder due to Lam and Pylyavskyy [LP12]. While in typical geometric or probabilistic applications the conductances are always positive real numbers, the algebraic nature of the problem leads us to consider general non-zero complex conductances.

To give a more precise statement, we start by defining the space of biperiodic networks. There is a natural equivalence relation on networks, defined by certain local rearrangements of the graph and its conductances, which does not change the spectral transform. To define this equivalence relation, let us start by defining a zig-zag path. A *zig-zag path* on G is a path that alternately turns maximally left or right. A resistor network G is *minimal* [CdVGV96, CIM98] if any lifts of any two zig-zag paths to $\tilde{G}$ do not intersect more than once and any lift of a zig-zag path has no self intersections. Minimality is a mild assumption on networks since any network may be reduced to a minimal one by certain elementary moves without affecting its electrical properties. The *Newton polygon* of a minimal resistor network is the unique integral polygon whose primitive edges are given by the homology classes of zig-zag paths in cyclic order. Since zig-zag paths come in pairs related by reversing orientation, the Newton polygon of a network is always centrally symmetric.

There is a local rearrangement of resistor networks called a Y-Δ move that preserves all electrical properties outside the region where the rearrangement takes place (see Section 2.2 and Figure 10). We say that two minimal networks (G_1, c_1) and (G_2, c_2) are *topologically equivalent* if there is a sequence of Y-Δ moves that takes the underlying graph G_1 to the graph G_2. Goncharov and Kenyon [GK13] showed that topological equivalence classes of networks are classified by centrally symmetric convex integral polygons. In other words, associated with any centrally symmetric convex integral polygon N is a minimal resistor network with Newton polygon N, and any two minimal resistor networks with Newton polygon N are related by a sequence of Y-Δ transformations.

Two networks (G_1, c_1) and (G_2, c_2) are *electrically equivalent* if there is a sequence of Y-Δ moves that takes the network (G_1, c_1) to the network (G_2, c_2). Goncharov and Kenyon [GK13] constructed the *resistor network cluster variety* $\mathcal{R}_N$ parameterizing the electrical equivalence classes of resistor networks that have Newton polygon N as follows: A centrally symmetric integral polygon N determines a finite collection of minimal resistor networks whose Newton polygon is N, related by Y-Δ transformations. To each minimal resistor network G is associated a complex torus $(\mathbb{C}^\times)^{\#\text{ edges of } G - 1}$, which parameterizes conductance functions on G. A Y-Δ transformation $G_1 \rightsquigarrow G_2$ induces a birational map between the complex tori associated with G_1 and G_2. The space $\mathcal{R}_N$ is obtained by gluing the complex tori for all graphs with Newton polygon N using these birational maps.

Goncharov and Kenyon further showed that $\mathcal{R}_N$ can be identified with an isotropic subvariety of an algebraic integrable system $\mathcal{X}_N$ associated with the dimer model in the torus. Let $\mathcal{S}_N$ be the moduli space of triples (C, S, ν), where C is an algebraic curve in $(\mathbb{C}^\times)^2$ defined by a Laurent polynomial $P(z, w)$ with Newton polygon N, S is a degree g effective divisor on C (where $g = \#$ interior lattice points in N = genus of C) and ν is a parameterization of the points at infinity of C. Kenyon and Okounkov [KO06] constructed a map $\mathcal{X}_N \to \mathcal{S}_N$ called the spectral transform. Fock [Foc15] showed that the spectral transform is birational and constructed an explicit inverse map using theta functions on the Jacobian variety of C.

For a biperiodic planar network, we can use the Laplacian to construct a *spectral transform* $\mathcal{R}_N \to \mathcal{S}_N$, where $\mathcal{S}_N$ is defined as in the previous paragraph, but with the divisor S now of degree $g = \#$ interior lattice points in $N - 1$. Let $\mathcal{S}'_N \subset \mathcal{S}_N$ be the subspace where $P(z, w)$ satisfies

1. $P(1, 1) = 0$ and the point $(1, 1)$ is a node;
2. The map $\sigma : (z, w) \mapsto (\frac{1}{z}, \frac{1}{w})$ is an involution on C,

and the divisor S satisfies

$$S + \sigma(S) - q_1 - q_2 = K_{\widehat{C}} \text{ in } \operatorname{Pic}^{2g-2}(\widehat{C}), \tag{1}$$

where $\widehat{C}$ is the normalization of C, g is the geometric genus of $\widehat{C}$, q_1, q_2 are the points in the fiber of the node at $(1, 1)$ and $K_{\widehat{C}}$ is the canonical divisor class on $\widehat{C}$. The appearance of the space $\mathcal{S}'_N$ is not surprising, since it has been studied in connection with the discrete BKP equation [DJKM82, Dol07], and the discrete BKP equation is related to the Y$-\Delta$ move by a change of coordinates ([GK13, Section 5.3.1]). Geometrically, the condition (1) satisfied by the divisor S

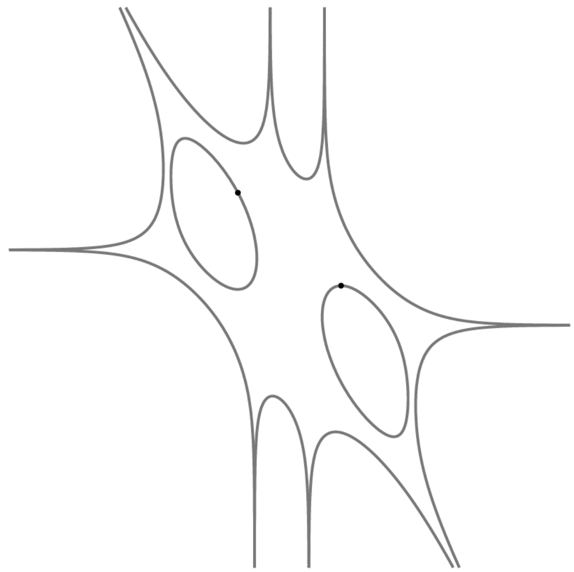

Figure 1: The standard divisor S on the amoeba of the spectral curve.

means that it lies in a translation of a certain subvariety of the Jacobian of $\widehat{C}$, called the Prym variety (cf. Proposition 6.6).

Our main result is the following complete classification of biperiodic planar resistor networks in terms of their spectral data:

Theorem 1.1 (cf. Theorem 6.18)**.** *The spectral transform $\mathcal{R}_N \to \mathcal{S}'_N$ is birational.*

The most difficult part of the proof of Theorem 1.1, and the main new contribution of this paper, lies in showing that the spectral divisor satisfies (1). Along the way, we provide an explicit description of oriented cycle-rooted spanning forests of G (abbreviated to OCRSFs hereafter) whose homology classes are boundary lattice points of N (Theorem 3.2), analogous to results for dimers in [Bro12, GK13]. In particular, we see that every OCRSF corresponding to a boundary lattice point is a union of cycles (Corollary 3.3).

We give an explicit inverse of the spectral transform (see (26)) in terms of theta functions on the Prym variety, following the algebro-geometric construction of the B-quadrilateral lattice by Doliwa [Dol07].

In Fock's construction, the local transformation in the dimer model called the spider move, which is analogous to the Y-Δ move in networks, is described by an identity for theta functions on the Jacobian called Fay's trisecant identity. Analogously, we show that:

Theorem 1.2 (cf. Theorem 7.1)**.** *The Y-Δ transformation is described by Fay's quadrisecant identity* [Fay89] *(cf. Theorem 6.7) for theta functions on the Prym variety.*

The Y-Δ move involves subtraction free rational expressions, and therefore, the set of positive-real-valued points of the cluster variety is well defined, which we denote by $\mathcal{R}_N(\mathbb{R}_{\geq 0})$. This subspace is important for probabilistic applications. For a positive real valued point, the spectral data (C, S, ν) has the following additional properties (see [Ken19]):

1. C is a simple Harnack curve as in [Mik00]. Compact ovals (connected components) of C are in bijection with interior lattice points of N.

2. The oval corresponding to the origin is degenerated to a real node.

3. S has a point in each of the other compact ovals, called a standard divisor in [KO06].

Spectral curves of genus zero correspond to the isoradial networks studied in [Ken02]. In this case, the inverse spectral map recovers Kenyon's results expressing the conductances in terms of tangents, and Fay's quadrisecant identity reduces to the triple tangent identity. For a different generalization of isoradial networks to the case of the massive Laplacian on isoradial graphs, see [BdTR17].

Consider the map $C(\mathbb{C}) \to \mathbb{A}(C), (z, w) \mapsto (\log |z|, \log |w|) \subset \mathbb{R}^2$ from the $\mathbb{C}$-valued points of C to its amoeba $\mathbb{A}(C)$. For a simple Harnack curve, this map is a homeomorphism from the compact ovals to the boundaries of the holes of the amoeba, and therefore provides a way to depict the divisor S (see Figure 1 for an example, where the network is a 2×1 fundamental domain of the triangular lattice).

A sequence of Y-Δ moves that takes a graph G to itself gives rise to a birational automorphism (called a cluster modular transformation) of $\mathcal{R}_N$, where N is the Newton polygon of G. A cluster modular transformation provides a discrete integrable system on $\mathcal{R}_N$. For example, if we consider the honeycomb lattice, and do the Y-Δ move at the downward triangles, we obtain the cube recurrence studied by Carroll and Speyer ([HS10], see also [GK13] Section 6.3). We show that cluster modular transformations are linearized in the Prym variety of $\widehat{C}$ (Theorem 8.1). In the case of positive-real-valued conductances, we may view this as moving each point along the boundary of the corresponding hole in the amoeba.

Organization of the paper. In Section 2, we collect background information on resistor networks in the torus, mostly following [GK13] and [Ken19]. In Section 3, we construct all extremal OCRSFs. In Section 4, we construct the spectral transform and prove some of its basic properties. The technical Section 5 forms the heart of the paper, and in it we find the image of the spectral transform. In Section 6, we review results about the Jacobian and Prym varieties, and prove Theorem 1.1. In Section 7, we prove Theorem 1.2 relating the Y-Δ transformation to Fay's quadrisecant identity, and use this to show that cluster modular transformations are linearized in the Prym variety in Section 8. Finally, the appendix collects basic results about Riemann surfaces that we will use extensively in Sections 4 and 5.

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2 Background

In this section, we give an introduction to resistor networks in the torus and the line bundle Laplacian.

2.1 Resistor networks in the torus

Let $\mathbb{T}$ denote the topological torus. A *resistor network* is a pair (G, c) where

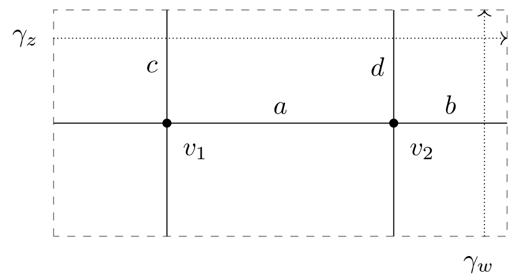


Figure 2: A resistor network in the torus (obtained by gluing opposite sides of the dashed rectangle) along with a conductance function. The loops γ_z, γ_w give a basis for $H_1(\mathbb{T}, \mathbb{Z})$.

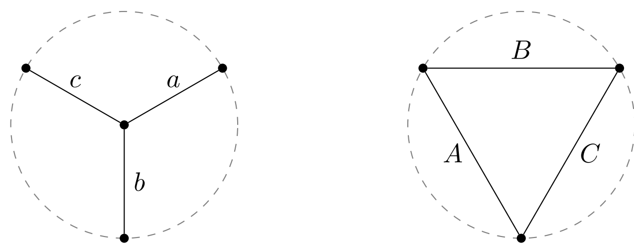


Figure 3: The Y-Δ move replaces a portion of the resistor network G that looks like the Y (on the left) with the Δ (on the right), or vice versa. The transformation rule for conductances is given by (5).

1. $G = (V, E, F)$ is a graph embedded in $\mathbb{T}$ such that its faces, i.e. connected components of $\mathbb{T} - G$, are topological disks, and
2. $c : E(G) \to \mathbb{C}^\times$ is a function defined modulo global multiplication by a nonzero complex number.

There is a local transformation of resistor networks called the Y$-\Delta$ move (see Figure 3). Two graphs G_1 and G_2 in $\mathbb{T}$ are said to be *topologically equivalent* if there is a sequence of Y$-\Delta$ moves that transform G_1 into G_2.

There is an invariant called the Newton polygon that classifies topological equivalence classes, which we now define. A *zig-zag path* in a resistor network G is an oriented path that alternately turns maximally right or left at each vertex. Zig-zag paths in G come in pairs with opposite orientations: Let $\overline{\alpha}$ denote the opposite zig-zag path of a zig-zag path α. We denote the set of zig-zag paths on G by $Z(G)$. The medial graph $G^\times$ of G is the graph obtained as follows:

1. Place a vertex t_e at the midpoint of each edge $e \in E(G)$, and
2. for $e, e' \in E(G)$, draw an edge between t_e and $t_{e'}$ if there is a face of G around which e and e' occur consecutively.

It is customary to represent a zig-zag path $e_1 \to e_2 \to \cdots \to e_n \to e_1$ using the path $t_{e_1}t_{e_2} \to t_{e_2}t_{e_1} \to \cdots \to t_{e_n}t_{e_1} \to t_{e_1}t_{e_2}$ in the medial graph.

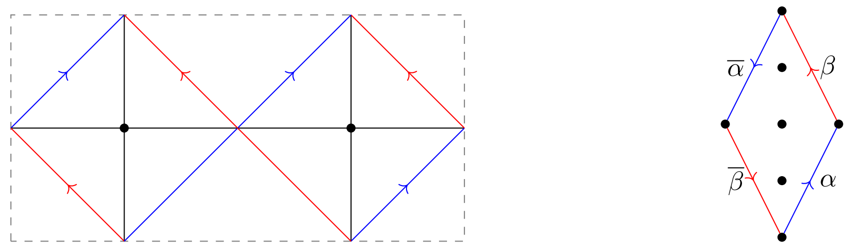


Figure 4: On the left is shown two of the four zig-zag paths of the resistor network in Figure 2. The other two zig-zag paths are obtained by reversing the orientation of the ones shown. On the right is its Newton polygon.

Let $\pi : \mathbb{R}^2 \to \mathbb{T}$ denote the universal cover of $\mathbb{T}$, and let $\widetilde{G} = \pi^{-1}(G)$ denote the biperiodic graph in the plane. We say that G is *minimal* if the lift of any zig-zag path to the universal cover $\mathbb{R}^2$ does not have self-intersections and if any lifts of two different zig-zag paths intersect at most once.

If G is a minimal graph, associated to each zig-zag path $\alpha \in Z(G)$ is its homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$. A convex polygon $N \subset H_1(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}^2$ is called *integral* if its vertices are contained in $H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$. By a *primitive edge vector* of N, we mean a vector contained in an edge of N and oriented in such a way that it is contained in the counterclockwise oriented boundary of N, so that its starting and ending points are integer lattice points. Since each edge of G is contained in two zig-zag paths that traverse the edge in opposite directions, $\sum_{\alpha \in Z(G)} [\alpha] = 0$. Therefore, there is a unique (modulo translation) convex integral polygon $N = N(G)$ in $H_1(\mathbb{T}, \mathbb{R})$ whose set of primitive edge vectors is $\{[\alpha] : \alpha \in Z(G)\}$, called the *Newton polygon* of G. The terminology will be justified in Section 2.4. A convex integral polygon N is said to be *centrally symmetric* if $(0, 0) \in N$ and is invariant under rotation by π. Since $[\overline{\alpha}] = -[\alpha]$, we can translate the Newton polygon of G so that it is centered at the origin to obtain a centrally symmetric polygon.

We define the *weight of a zig-zag path* α as follows: Suppose

$$\alpha = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} v_n \xrightarrow{e_n} v_1,$$

where α turns maximally right at v_1, v_3 etc, and maximally left at v_2, v_4 etc. Then

$$wt(\alpha) := \frac{\prod_{i \text{ odd}} c(e_i)}{\prod_{i \text{ even}} c(e_i)} \tag{2}$$

is the alternating product of conductances around α. Since left and right turns alternate, there is an equal number of factors in the numerator and denominator of (2). We also note that

$$wt(\overline{\alpha}) = wt(\alpha). \tag{3}$$

Example 2.1. For the resistor network (G, c) shown in Figure 2, there are four zig-zag paths (see the left hand side of Figure 4). In the basis $([\gamma_z], [\gamma_w])$ for $H_1(\mathbb{T}, \mathbb{Z})$ in Figure 2, the homology classes of the four zig-zag paths are

$$(2, -1), \quad (-1, -2), \quad (1, -2), \quad (1, 2).$$

The Newton polygon is shown on the right hand side of Figure 4.

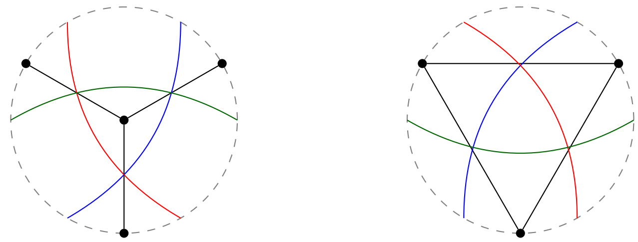


Figure 5: Correspondence between zig-zag paths in the Y and the Δ, which we can think of as sliding one of the zig-zag paths through the intersection point of the other two. Since the zig-zag paths are unchanged outside the dashed disk, their homology classes are invariant.

A Y$-\Delta$ move does not change the homology class of any zig-zag path (Figure 5), and therefore the Newton polygon is invariant under topological equivalence, so the Newton polygon is a well-defined function.

$$\{\text{Minimal torus graphs}\}/\text{topological equivalence} \xrightarrow{G\mapsto N(G)} \\ \{\text{Centrally symmetric convex integral polygons in } H_1(\mathbb{T},\mathbb{R})\}. \tag{4}$$

Theorem 2.2 ([GK13])**.** *The function (4) mapping a graph to its Newton polygon is a bijection.*

In other words, for each centrally symmetric convex integral polygon N in $H_1(\mathbb{T},\mathbb{R})$, there is a family of minimal resistor networks with Newton polygon N, and any two members of the family are related by Y-Δ moves.

2.2 The resistor network cluster variety

So far, we have only considered the underlying graph of a resistor network and not the conductance. In this section, we define a space parameterizing resistor networks with Newton polygon N, following [GK13].

Let N be a centrally symmetric convex integral polygon in $H_1(\mathbb{T},\mathbb{R})$. For a minimal resistor network G with $N(G) = N$, let

$$\mathcal{R}_G := \{c : E(G) \to \mathbb{C}^\times\}/\mathbb{C}^\times \cong (\mathbb{C}^\times)^{\#E(G)-1}$$

be the space of conductances on G. A Y-Δ move $s : G_1 \rightsquigarrow G_2$ induces a birational map $\mu^s : \mathcal{R}_{G_1} \dashrightarrow \mathcal{R}_{G_2}$ (we denote rational maps by $\dashrightarrow$), given in the notation of Figure 10 by

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c}. \tag{5}$$

Two minimal resistor networks (G_1, c_1) and (G_2, c_2) with Newton polygon N are said to be *electrically equivalent* if there is a sequence of Y$-\Delta$ moves transforming (G_1, c_1) into (G_2, c_2). Gluing the spaces $\mathcal{R}_G$ for all minimal G with Newton polygon N using the birational maps induced by Y$-\Delta$ moves, we obtain a space $\mathcal{R}_N$ parameterizing electrical equivalence classes of minimal resistor networks with Newton polygon N, called the *resistor network cluster variety.*

2.3 The line bundle Laplacian

In this section, we describe the line bundle Laplacian for a general resistor network (a graph G with a conductance function $c : E(G) \to \mathbb{C}^\times$, not necessarily embedded in $\mathbb{T}$), a variant of the discrete Laplacian, that captures additional topological information. In Section 2.4, we will specialize the construction to resistor networks in $\mathbb{T}$ and flat line bundles with connection.

A discrete *line bundle with connection* (L, ϕ) on a graph G is the data of:

1. A complex line $L_v \cong \mathbb{C}$ at each vertex v of G, and

2. An isomorphism $\phi(e) : L_v \to L_u$, called *parallel transport*, for each directed edge $e = v \to u$ such that $\phi(e) = \phi(\overline{e})^{-1}$, where $\overline{e} = u \to v$ denotes the edge e oriented in the opposite direction.

Two line bundles with connection (L, ϕ) and (L', ϕ') are said to be *isomorphic* or *gauge equivalent* if there exists isomorphisms $\psi(v) : L_v \to L'_v$ such that for all directed edges $e = v \to u$ of G, the following diagram commutes

$$\begin{array}{ccc} L_v & \xrightarrow{\phi(e)} & L_u \\ {\scriptstyle\psi(v)}\big\downarrow & & \big\downarrow{\scriptstyle\psi(u)} \\ L'_v & \xrightarrow{\phi'(e)} & L'_u \end{array} .$$

If $\gamma = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} v_n \xrightarrow{e_n} v_1$ is an oriented cycle in G, the *monodromy* $m(\gamma)$ of (L, ϕ) around γ is the composition

$$L_{v_1} \xrightarrow{\phi(e_1)} L_{v_2} \xrightarrow{\phi(e_2)} \cdots \xrightarrow{\phi(e_{n-1})} L_{v_n} \xrightarrow{\phi(e_n)} L_{v_1}$$

of the parallel transports around γ. Using the identification $\mathrm{GL}(L_{v_1}) \cong \mathbb{C}^\times$, we consider $m(\gamma)$ to be a nonzero complex number. Note that this complex number does not change if we use another v_i to define it instead of v_1. The *moduli space of line bundles with connection on G modulo gauge equivalence* is denoted by $\mathcal{L}_G$.

Let (G, c) be a resistor network and let (L, ϕ) be a line bundle with connection on G. Let $V(G)$ denote the set of vertices of G. The *line bundle Laplacian* is the linear operator

$$\Delta : \bigoplus_{v \in V(G)} L_v \to \bigoplus_{v \in V(G)} L_v$$
$$\Delta(f)(v) := \sum_{e: u \to v} c(e)(f(v) - \phi(e) f(u)),$$

where the sum is over all directed edges of G oriented towards v. An *oriented cycle-rooted spanning forest* (OCRSF) F of G is a collection of edges of G such that each connected component of F has the same number of vertices and edges (so that each connected component has a unique cycle), along with a choice of orientation for each cycle in F. The weight of an *OCRSF F* is defined to be $wt(F) = \prod_{e \in F} c(e)$. The following result generalizes Kirchhoff's matrix tree theorem to the line bundle Laplacian.

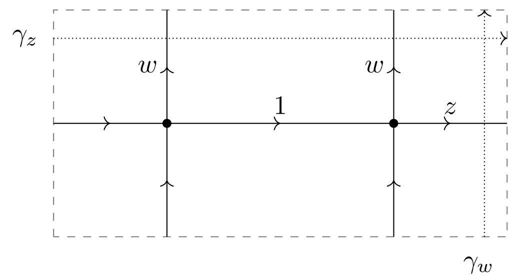


Figure 6: The flat line bundle with connection (L, ϕ) associated to $(z, w) \in (\mathbb{C}^\times)^2$ for the resistor network from Figure 2. The edges have been given an arbitrary orientation and the number next to each directed edge indicates the connection in that direction, while the connection for the edge in the other direction is the reciprocal.

Theorem 2.3 (Kenyon, 2010 [Ken11])**.** *Let (G, c) be a resistor network and let (L, ϕ) be a line bundle with connection on G.*

$$\det \Delta = \sum_{OCRSFs\ F} wt(F) \prod_{cycles\ \eta \in F} (1 - m(\eta)),$$

where $m(\eta)$ is the monodromy of (L, ϕ) along the cycle η.

2.4 Flat line bundles with connection in $\mathbb{T}$

We now return to the case of minimal resistor networks in $\mathbb{T}$. In this section, we give a more explicit coordinate description of the line bundle Laplacian.

Let (G, c) be a minimal resistor network in $\mathbb{T}$. A line bundle with connection on G is called *flat* if the monodromy around the boundary of any face of G is trivial. Let $\mathcal{L}_G^{\text{flat}} \subset \mathcal{L}_G$ be the subspace of flat connections. The monodromies around loops in G give rise to isomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_G^{\text{flat}} & \hookrightarrow & \mathcal{L}_G \\ \downarrow \cong & & \downarrow \cong \\ H^1(\mathbb{T}, \mathbb{C}^\times) & \hookrightarrow & H^1(G, \mathbb{C}^\times), \end{array}$$

where the bottom arrow comes from the embedding $G \hookrightarrow \mathbb{T}$. Therefore a flat line bundle with connection is the same thing as a cohomology class in $H^1(\mathbb{T}, \mathbb{C}^\times)$. We give a coordinate description of the inverse map $(\mathbb{C}^\times)^2 \cong H^1(\mathbb{T}, \mathbb{C}^\times) \xrightarrow{\cong} \mathcal{L}_G^{\text{flat}}$.

Let R be a fundamental rectangle for $\mathbb{T}$, so that $\mathbb{T}$ is obtained by gluing together opposite sides of R. We label the curves in $\mathbb{T}$ forming the sides of R by γ_z, γ_w, oriented as shown in Figure 6, so that $([\gamma_z], [\gamma_w])$ is a basis for $H_1(\mathbb{T}, \mathbb{Z})$. For $(z, w) \in (\mathbb{C}^\times)^2$, we define a flat line bundle with connection (L, ϕ) on G as follows:

1. Let $L_v \cong \mathbb{C}$ be a complex line at each vertex v of G, and

2. For a directed edge $e : u \to v$ of G, let $\phi(e) : L_u \to L_v$ be defined as multiplication by

$$z^{(e,\gamma_w)_{\mathbb{T}}} w^{(e,-\gamma_z)_{\mathbb{T}}} \in \mathbb{C}^\times,$$

where $(\cdot,\cdot)_{\mathbb{T}}$ is the intersection pairing in $\mathbb{T}$.

For this flat line bundle with connection, let $\Delta(z,w)$ denote the line bundle Laplacian. We can rephrase Theorem 2.3 as

$$\det \Delta(z,w) = \sum_{\text{OCRSFs } F} wt(F) \prod_{\text{cycles } \eta \in F} (1 - z^{i(\eta)} w^{j(\eta)}), \tag{6}$$

where $(i(\eta), j(\eta)) \in \mathbb{Z}^2$ is the homology class of the cycle η in the basis $([\gamma_z],[\gamma_w])$ for $H_1(\mathbb{T},\mathbb{Z})$. The Laurent polynomial $P(z,w) := \det \Delta(z,w)$ is called the *characteristic polynomial*. The curve $C_0 := \{(z,w) \in (\mathbb{C}^\times)^2 : P(z,w) = 0\}$ is called the *(open) spectral curve*.

Remark 2.4. Since the line bundle with connection is flat, for an OCRSF F with a *topologically trivial cycle*, that is, a cycle η such that $[\eta] = 0$ in $H_1(\mathbb{T},\mathbb{Z})$), we have

$$\prod_{\text{cycles } \eta \in F} (1 - z^{i(\eta)} w^{j(\eta)}) = 0,$$

and therefore such OCRSFs do not contribute to $P(z,w)$. If F has no topologically trivial cycles, since two distinct cycles in F cannot intersect, if η is a cycle in F, every cycle in F has homology class $\pm[\eta]$.

The *Newton polygon of* $P(z,w)$ is defined as

$$N(P(z,w)) = \text{Convex-hull}\{(i,j) \in \mathbb{Z}^2 : \text{coefficient of } z^i w^j \text{ is non-zero in } P(z,w)\}.$$

Proposition 2.5 ([GK13])**.** *Let (G,c) be a minimal resistor network in $\mathbb{T}$ with Newton polygon N. Let $P(z,w)$ denote the characteristic polynomial. Then, $N(P(z,w)) = N$.*

This proposition justifies the name Newton polygon for $N(G)$.

Example 2.6. Let us compute the Laplacian and the characterisitic polynomial for the resistor network in Figure 2. The line bundle Laplacian is given by the matrix

$$\Delta(z,w) = \begin{bmatrix} a+b+c(2-w-1/w) & -a-bz \\ -a-b/z & a+b+d(2-w-1/w) \end{bmatrix}. \tag{7}$$

Therefore

$$P(z,w) = cd\left((1-w)^2 + \left(1-\frac{1}{w}\right)^2\right) + ab\left((1-z) + \left(1-\frac{1}{z}\right)\right) + (ac+bc+ad+bd)\left((1-w) + \left(1-\frac{1}{w}\right)\right),$$

enumerating the 12 OCRSFs of this resistor network. Moreover,

$$N(P(z,w)) = \text{Convex-hull}\{(0,0),(1,0),(-1,0),(0,1),(0,-1),(0,2),(0,-2)\}$$

coincides with $N(G)$ in Figure 4.

Proposition 2.7. *The characteristic polynomial $P(z,w)$ has the following properties:*

1. $P(z,w) = P(\frac{1}{z}, \frac{1}{w})$*;*
2. $(1,1) \in C_0$*;*
3. *The point* $(1,1)$ *is a singular point of* C_0*.*

Proof.

1. $P(z,w) = P(\frac{1}{z}, \frac{1}{w})$ follows from $\Delta(z,w) = \Delta(\frac{1}{z}, \frac{1}{w})^T$.
2. $P(1,1) = 0$ follows from the observation that $\Delta(z,w)$ has nonzero kernel at $(1,1)$; constant functions are discrete harmonic.
3. Differentiating the expression (6) for $P(z,w)$, we see that

$$\frac{\partial P(1,1)}{\partial z} = \frac{\partial P(1,1)}{\partial w} = 0,$$

hence $(1,1)$ is a singular point.

□

3 Extremal OCRSFs

The goal of this section is to give a characterization of extremal OCRSFs (cf. Theorem 3.2). This result will only be used in the proof of Theorem 5.7.

An OCRSF $F^\vee$ on $G^\vee$ is *dual* to an OCRSF F on G if no edge of $F^\vee$ crosses an edge of F. It is easy to see that $F^\vee$ has the same number of cycles as F and each cycle has homology class $\pm[\eta]$, where η is any cycle in F. An OCRSF F has 2^k duals where k is the number of cycles in F, one for each choice of orientation of the dual cycles.

Given a pair $(F, F^\vee)$ of dual OCRSFs, define its weight to be $wt(F, F^\vee) := wt(F)$. To $(F, F^\vee)$ we associate a homology class,

$$[(F, F^\vee)] := \frac{1}{2} \sum_{\text{cycles } \eta \text{ in } F \cup F^\vee} [\eta] \in H_1(\mathbb{T}, \mathbb{Z}).$$

Then the Newton polygon of the resistor network is

$$N = \text{Convex-hull}\{[(F, F^\vee)] \in H_1(\mathbb{T}, \mathbb{Z}) : (F, F^\vee) \text{ is a pair of dual OCRSFs}\}.$$

The map $(F, F^\vee) \mapsto [(F, F^\vee)]$ associates to each pair of dual OCRSFs an integer lattice point in the Newton polygon.

We say that a pair of dual OCRSFs $(F, F^\vee)$ is *external* if $[(F, F^\vee)]$ is a boundary lattice point of N. It is *extremal* if $[(F, F^\vee)]$ is a vertex of N. We note that if $(F, F^\vee)$ is external, then the orientations of F and $F^\vee$ are uniquely determined by the homology class $[(F, F^\vee)]$, and $[F] = [F^\vee] = [(F, F^\vee)]$. This observation allows us to define external and extremal OCRSFs on G, rather than pairs of dual OCRSFs.

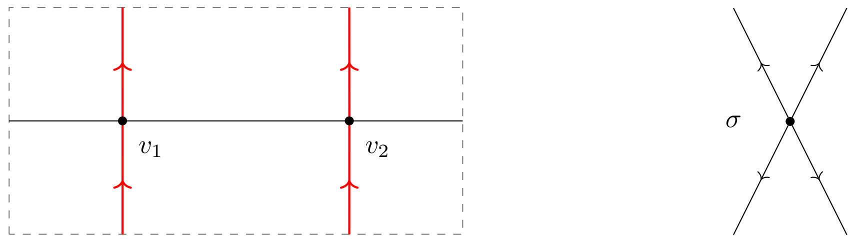


Figure 7: On the left (in red) is the extremal OCRSF $F_{(0,2)}$ corresponding to the resistor newtork in Figure 2. On the right the zig-zag fan Σ along with the 2-dimensional cone σ corresponding to the vertex $(0, 2)$ of the Newton polygon on right side of Figure 4.

3.1 Local and global zig-zag fans

Following [Bro12], for a vertex $v \in G$, we define the *local zig-zag fan* Σ_v at v as the complete fan of strongly convex rational polyhedral cones in $H_1(\mathbb{T}, \mathbb{R})$ whose rays are generated by homology classes of zig-zag paths through v that turn maximally right at v.

The fan Σ whose rays are generated by the homology classes of all zig-zag paths on G is called the *global zig-zag fan* of G.

Remark 3.1. The global zig-zag fan is not the dual fan of N, but it is isomorphic to it.

We have the natural map of fans $i_v : \Sigma \to \Sigma_v$ for each $v \in G$. If σ is a two-dimensional cone in Σ, $i_v(\sigma)$ is contained in a unique two-dimensional cone in Σ_v, which we denote by σ_v. σ_v determines a unique edge $e \in E$ adjacent to v that is oriented away from v: e is the edge that contains the two zig-zag paths corresponding to the rays of σ_v. Let F_{σ_v} be the 1-chain that is 1 on e, -1 on $-e$ and 0 on all other edges. We define

$$F_\sigma := \sum_{v \in V(G)} F_{\sigma_v}.$$

To a zig-zag path $\alpha \in Z$ we associate a 1-chain ω_α that is 1 on edges e in α that are oriented in the same direction as α and 0 on edges not in α. If F is external, $[F]$ lies on an edge E of N, which corresponds to a family of zig-zag paths $\{\alpha_k\}$. Let $E = V_1V_2$, where V_1, V_2 are vertices of N such that V_2 is the vertex after V_1 when the boundary of N is traversed counterclockwise.

The following theorem explicitly describes all external OCRSFs.

Theorem 3.2. *$F_V := F_\sigma$ is the unique extremal OCRSF on G such that $[F_V]$ is the vertex V of N that is dual to σ.*

Let A be a subset of the family of zig-zag paths $\{\alpha_k\}$ corresponding to E. The external OCRSFs on E are of the form

$$F_A := F_{V_1} + \sum_{\alpha_k \in A} \omega_{\alpha_k}.$$

In particular, $F_{V_2} = F_{V_1} + \sum_k \omega_{\alpha_k}$, and the number of OCRSFs corresponding to a boundary lattice point of N is a binomial coefficient.

We also need the following result later.

Corollary 3.3. *Every external OCRSF is a disjoint union of cycles.*

Both Theorem 3.2 and Corollary 3.3 will be proved in Section 3.2.

Example 3.4. Let us compute the extremal OCRSF of the network in Figure 2 corresponding to the vertex $(0,2)$ of its Newton polygon. The global zig-zag fan Σ has rays generated by $(-1,2),(-1,-2),(1,-2),(1,2)$ (shown on the left side of Figure 7), and coincides with the local zig-zag fans $\Sigma_{v_1},\Sigma_{v_2}$. Let us consider v_1. σ is the 2-dimensional cone with rays generated by $(-1,2)$ and $(-1,-2)$. Since $i_{v_1}:\Sigma\to\Sigma_{v_1}$ is the identity map, $\sigma_{v_1}=\sigma$. Therefore $F_{\sigma_{v_1}}$ is the 1-chain that is 1 on the edge with conductance c, oriented upwards. Similarly $F_{\sigma_{v_2}}$ is the edge with conductance d oriented upwards. $F_{(0,2)}$ is the OCRSF given by the union of these two oriented edges (Figure 7). As we expect from Corollary 3.3, it is a union of (two) cycles.

3.2 Proof of Theorem 3.2

While its possible to prove Theorem 3.2 directly, it is easier to use Temperley's bijection to relate it to corresponding statements about the dimer model. The results of this section are not used anywhere else in the paper, and therefore may be skipped on a first reading. Let Γ be a *bipartite* surface graph on $\mathbb{T}$, that is the vertices of Γ are colored black or white, and each edge of Γ is incident to a vertex of each color.

A *dimer cover* (or *perfect matching*) of Γ is a collection of edges of Γ such that every vertex is adjacent to a unique edge in the collection. A dimer cover M on Γ gives a 1-chain ω_M on Γ. If M_0 is another dimer cover, $\omega_M-\omega_{M_0}$ is a 1-cycle and therefore determines a homology class in $H_1(\Gamma,\mathbb{Z})$. Under the projection $H_1(\Gamma,\mathbb{Z})\to H_1(\mathbb{T},\mathbb{Z})$, we obtain a homology class $[M]\in H_1(\mathbb{T},\mathbb{Z})$. The *Newton polygon* of Γ is

$$N:=\text{Convex-hull}\{[M]\in H_1(\mathbb{T},\mathbb{Z}):M\text{ is a dimer cover}\}.$$

N depends on the choice of reference dimer cover M_0. Changing the reference matching corresponds to translating the polygon N. $M\mapsto[M]$ gives a well defined map from the set of dimer covers to the integer lattice points in N.

3.2.1 Zig-zag paths on bipartite graphs and minimality

A *zig-zag path* on a bipartite torus graph Γ is a path that turns maximally right at black vertices and maximally left at white vertices. Let us denote by $Z(\Gamma)$ the set of all zig-zag paths in Γ. We say that Γ is *minimal* if in the universal cover $\widetilde{\Gamma}$, zig-zag paths have no self intersections and no pairs of zig-zag paths oriented in the same direction meet twice.

Suppose Γ is a minimal bipartite graph on a torus. Each path $\alpha\in Z(\Gamma)$ gives us a homology class $[\alpha]\in H_1(\mathbb{T},\mathbb{Z})$ which is an integral pimitive vector on a side of the Newton polygon N. The zig-zag paths taken in cyclic order correspond to cyclically ordered primitive integral vectors in the boundary of the Newton polygon. Therefore an edge of N corresponds to a family of zig-zag paths, each with homology class equal to the primitive integral edge vector of the edge.

3.2.2 Temperley's bijection on the torus

Associated to G is a bipartite graph Γ_G obtained by superposing G and its dual graph $G^\vee$. The vertices and faces of G become the black vertices of Γ_G and the edges of G become the white vertices of Γ_G. Applying Euler's formula on $\mathbb{T}$ to G we see that Γ_G has equal number of white and black vertices.

Let G be a resistor network and let Γ_G be the associated bipartite graph.

Lemma 3.5 (Goncharov and Kenyon, 2012 [GK13]). *The Newton polygon N of the resistor network G coincides with the Newton polygon of the dimer model on Γ_G. Moreover, there is a canonical homology-class-preserving bijection between $Z(G)$ and $Z(\Gamma_G)$.*

Given a pair of dual OCRSFs $(F, F^\vee)$ on G, we can construct a dimer cover $M_{(F,F^\vee)}$ on Γ_G using the rule: The oriented edge $e = uv$ is in $F \cup F^\vee$ if and only if the edge ue is in M_F.

Theorem 3.6 (Temperley's bijection on torus; Kenyon, Propp and Wilson, 2000 [KPW00]). *Let (G, c) be a resistor network on a torus. $(F, F^\vee) \mapsto M_{(F,F^\vee)}$ is a bijection from pairs of dual OCRSFs on G to dimer covers on Γ_G such that $[(F, F^\vee)] = [M_{(F,F^\vee)}]$.*

Note that there is a canonical bijection between Z and Z_{Γ_G} that preserves homology classes.

3.2.3 External dimer covers

In this section, we collect some results about dimer covers from [Bro12, GK13]. Let Γ be a minimal bipartite graph on a torus. We say that a dimer cover M is *extremal* if $[M]$ is a vertex of the Newton polygon. If b is any black vertex in Γ, we define the *local zig-zag fan* Σ_b at b to be the complete fan of strongly convex rational polyhedral cones in $H_1(\mathbb{T}, \mathbb{Z})$ whose rays are generated by homology classes of those zig-zag paths in Γ that contain b.

The *global zig-zag fan* of Γ is the fan whose rays are generated by the homology classes of all zig-zag paths on Γ. The identity map in $H_1(\mathbb{T}, \mathbb{Z})$ defines a map of fans $i_b : \Sigma \to \Sigma_b$. If σ is any two dimensional cone in Σ, $i_b(\sigma)$ is contained in a unique two dimensional cone in Σ_b which we call σ_b. σ_b corresponds to a unique edge wb incident to b, given by the intersection of the two zig-zag paths through b whose rays in Σ_b form the boundary of σ_b. Define the 1-chain $\omega(\sigma_b)$ to be 1 on the edge wb and 0 on all other edges. Define

$$\omega(\sigma) = \sum_{b \in V(\Gamma) \text{ black}} \omega(\sigma_b).$$

Two dimensional cones in Σ are in bijection with vertices of the Newton polygon: If σ is a two dimensional cone in Σ, let E_1 and E_2 be the edges of N whose associated rays form the boundary of σ in Σ. Then E_1 and E_2 occur in cyclic order and therefore there is a vertex V between them in N.

Lemma 3.7 (Broomhead, Goncharov-Kenyon, 2012 [Bro12, GK13]). *$\omega_V := \omega(\sigma)$ is the unique extremal dimer cover associated to the vertex V of N that corresponds to σ.*

We say that a dimer cover M is *external* if $[M]$ is a boundary lattice point of N. To a zig-zag path α we associate a 1-chain ω_α that is 1 on edges e in α that are oriented the same way as α and 0 on edges not in α. If M is external, $[M]$ lies on an edge E of N, which corresponds to a family of zig-zag paths $\{\alpha_k\}$. Let $E = V_1V_2$, where V_1, V_2 are vertices of N such that V_2 is the vertex after V_1 when the boundary of N is traversed counterclockwise.

Lemma 3.8 (Broomhead, Goncharov-Kenyon, 2012 [Bro12, GK13]). *Let A be a subset of the family of zig-zag paths $\{\alpha_k\}$ corresponding to E. The external dimer covers on E are of the form*

$$\omega_A := \omega_{V_1} + \sum_{\alpha_k \in A} \omega_{\alpha_k}.$$

In particular, $\omega_{V_2} = \omega_{V_1} + \sum_k \omega_{\alpha_k}$, and the number of dimer covers corresponding to a boundary lattice point of N is a binomial coefficient.

Proof of Theorem 3.2. Follows immediately from Temperley's bijection (Theorem 3.6), Lemmas 3.7 and 3.8, and the canonical bijection between zig-zag paths on G and Γ_G. □

Proof of Corollary 3.3. Suppose F_σ is an external OCRSF and let v be a vertex of G. By construction, there is a single outgoing edge from v. We show that there is also a single incoming edge. Consider the fan $-\Sigma_v$ whose rays are generated by homology classes of zig-zag paths that turn maximally left at v and let $i'_v : \Sigma \to -\Sigma_v$ be the natural map. $i'_v(\sigma)$ is contained in a unique two dimensional cone σ'_v which corresponds to a unique edge e oriented towards v. Define the 1-chain F'_{σ_v} to be 1 on e and 0 on all other edges and define the 1-chain

$$F'_\sigma := \sum_{v \in V(G)} F'_{\sigma_v}.$$

Let $e = uv$ be an edge in G and let α_1 and α_2 be the two zig-zag paths through e that turn maximally left at v. Then α_1 and α_2 turn maximally right at u and therefore we have $\sigma'_v = \sigma_u$ which implies $F'_{\sigma_v} = F_{\sigma_u}$. Summing over all vertices, we get $F'_\sigma = F_\sigma$. It is clear from the definition of F'_σ that every vertex has a unique incoming edge. It follows that F_σ is a union of cycles.

By Theorem 3.2, every external OCRSF is obtained from an extremal OCRSF F_V by adding cycles corresponding to some zig-zag paths and therefore is also a union of cycles.

□

4 The spectral transform

In this section, we define the spectral transform. To use the theory of divisors, we need to be working with a compact Riemann surface/proper smooth algebraic curve. This requires dealing with two technical issues first:

1. The open spectral curve C_0 is not compact. The standard way to fix this is to compactify it by taking the closure of C_0 in the toric surface associated to the Newton polygon.
2. The open spectral curve has a node, which we will resolve by a normalization [Vak17, Section 9.7].

We try to not assume much prior knowledge of toric surfaces, but do assume that the reader is familiar with the theory of compact Riemann surfaces, and line bundles and divisors on them (also see the Appendix).

4.1 Toric surfaces

We give an informal introduction to the toric surface X_N associated to a polygon N that is sufficient for our purposes and refer the reader to [CLS11,Ful93] for the detailed constructions. A *toric surface* is an algebraic surface X (over $\mathbb{C}$) that contains the torus $(\mathbb{C}^\times)^2$ as a dense open subvariety, such that the action of $(\mathbb{C}^\times)^2$ on itself by multiplication extends to all of X. A convex integral polygon N defines a projective toric surface X_N, that is a toric surface embedded in a projective space as a closed subvariety. In particular, X_N is compact (proper). The geometry of X_N is determined by the combinatorics of the polygon N. The complement of $(\mathbb{C}^\times)^2$ in X_N is a union on $\mathbb{P}^1$s, called the *lines at infinity* of X_N, that intersect according to the combinatorics of N:

1. Each edge E of N corresponds to a $\mathbb{P}^1 \subset X_N$, which we denote by D_E;
2. $X_N - (\mathbb{C}^\times)^2 = \bigcup_E D_E$;
3. If two edges E_1 and E_2 have a vertex of N in common, then $D_{E_1} \cap D_{E_2}$ is a single point;
4. If E_1 and E_2 do not have a vertex of N in common, then D_{E_1} and D_{E_2} are disjoint.

We give two examples of toric surfaces that can be understood very explicitly and illustrate the general theory.

Example 4.1. If the Newton polygon is Convex-hull$\{(0,0),(1,0),(0,1)\}$, then the toric surface X_N is $\mathbb{P}^2$ with the projective embedding given by the identity map. Let

$$\mathbb{P}^2 = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^\times$$

be the quotient construction of $\mathbb{P}^2$ and let $[x_0, x_1, x_2]$ denote the homogeneous coordinates. The embedding of the torus is

$$\begin{aligned}(\mathbb{C}^\times)^2 &\hookrightarrow \mathbb{P}^2\\ (z,w) &\mapsto [1:z:w].\end{aligned}$$

Therefore

$$\mathbb{P}^2 - (\mathbb{C}^\times)^2 = D_{E_0} \cup D_{E_1} \cup D_{E_2},$$

where $D_{E_i} = \{[x_0 : x_1 : x_2] : x_i = 0\}$ is a $\mathbb{P}^1$. These are the three axes of $\mathbb{P}^2$, and any two of them intersect in a point. For example,

$$D_{E_0} \cap D_{E_1} = \{[x_0 : x_1 : x_2] : x_0 = x_1 = 0\} = \{[0:0:1]\}.$$

Example 4.2. If $N = $ Convex-hull$\{(0,0),(1,0),(0,1),(1,1)\}$ is the unit square, the associated toric surface X_N is $\mathbb{P}^1 \times \mathbb{P}^1$. Let $([x_0 : x_1],[x_2 : x_3])$ denote the homogeneous coordinates. The projective embedding of X_N is the Segre embedding

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3\\ ([x_0 : x_1],[x_2 : x_3]) &\mapsto [x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3].\end{aligned}$$

The embedding of the torus is

$$\begin{aligned}(\mathbb{C}^\times)^2 &\hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1\\ (z,w) &\mapsto ([1:z],[1:w]),\end{aligned}$$

and

$$\mathbb{P}^1 \times \mathbb{P}^1 - (\mathbb{C}^\times)^2 = D_{E_0} \cup D_{E_2} \cup D_{E_3} \cup D_{E_4},$$

where

$$\begin{aligned}D_{E_0} &= [1:0] \times \mathbb{P}^1, \quad D_{E_2} = [0:1] \times \mathbb{P}^1,\\ D_{E_1} &= \mathbb{P}^1 \times [1:0], \quad D_{E_3} = \mathbb{P}^1 \times [0:1].\end{aligned}$$

It is easy to check that they intersect according to the combinatorics of the square, for example $D_{E_0} \cap D_{E_1} = ([1:0],[0:1]), D_{E_0} \cap D_{E_2} = \varnothing$ etc.

4.2 Curves in toric surfaces

Let $C_0 \subset (\mathbb{C}^\times)^2$ be a curve defined by a Laurent polynomial $P(z,w)$ with Newton polygon N. Let X_N be the toric surface associated to N. Taking the closure of C_0 in X_N, we get a compact curve C such that $C_0 = C \cap (\mathbb{C}^\times)^2$.

We record the following properties of a generic curve C in X_N with Newton polygon N.

1. The curve C has genus equal to # interior lattice points in N [CLS11, Proposition 10.5.8], generalizing the degree-genus formula for $\mathbb{CP}^2$ [Vak17, (18.6.6.1)].

2. The curve C meets the line at infinity D_E in $|E|$ points (counted with multiplicity), called the *points at infinity* of C. Here $|E|$ denotes the *integral length* of E, that is the number of primitive integral vectors in E. For $\mathbb{CP}^2$, this follows from Bezout's theorem [Vak17, Exercise 18.6.K].

For an open spectral curve C_0, its closure C is called the *spectral curve*. For an edge E of N, let $Z(E) \subset Z$ denote the set of zig-zag paths whose homology classes are primitive edge vectors in E. Then we have $|E| = \#Z(E) = \#C \cap D_E$. A *parameterization* ν *of the points at infinity* of C by zig-zag paths is a collection $\{\nu_E\}$ of bijections $\nu_E : Z(E) \to C \cap D_E$ as E varies over the set of edges of N.

A *divisor* S on a curve C is a formal linear-combination $S = \sum_{i=1}^n a_i p_i$ of points p_i of C, where $a_i \in \mathbb{Z}$. If $a_i \geq 0, i \in [n]$, then S is called an *effective* divisor. The sum $\sum_{i=1}^n a_i$ is called the degree of S and denoted $\deg\ S$.

4.3 The spectral transform

Let G be a minimal resistor network associated to N and let v be a vertex of G. Let $\mathcal{S}_N$ be the moduli space of triples (C, S, ν) such that

1. C is a curve in X_N with Newton contained in N;

2. S is a degree g effective divisor on C;

3. ν is a parameterization of the points at infinity of C by zig-zag paths of G.

Let $i : C_0 \hookrightarrow (\mathbb{C}^\times)^2$ denote the inclusion. The line bundle Laplacian is a map of trivial vector bundles on $(\mathbb{C}^\times)^2$:

$$\bigoplus_{v \in V(G)} \mathcal{O}_{(\mathbb{C}^\times)^2} \xrightarrow{\Delta(z,w)} \bigoplus_{v \in V(G)} \mathcal{O}_{(\mathbb{C}^\times)^2} \tag{8}$$

Suppose the conductance is generic, and let $\mathcal{L} = \text{coker}\ \Delta(z,w)$. Since C_0 is the locus where $P(z,w) = \det \Delta(z,w) = 0$, we have the following:

1. If $(z,w) \in C_0 \setminus (1,1)$ and the conductance is generic, then C_0 is smooth at (z,w). In this case, the cokernel of $\Delta(z,w)$ is one-dimensional [CT79, Theorem 2.2];

2. The cokernel at the singular point $(1,1)$ is the vector space of discrete harmonic functions on G. This space is one-dimensional because, by the maximum principle, the only harmonic functions are the constant functions.

3. If $(z,w) \notin C_0$, then $\Delta(z,w)$ is nonsingular and the cokernel is 0.

Therefore, we see that $\mathcal{L}$ is not a vector bundle on $(\mathbb{C}^\times)^2$, but a coherent sheaf that is supported on C_0, and the fibers over C_0 are all one-dimensional.

Lemma 4.3. *For a generic conductance, the restriction $i^*\mathcal{L}$ of $\mathcal{L}$ to C_0 is a line bundle on C_0.*

Proof. Since C_0 is integral (i.e., irreducible and reduced) for a generic conductance and $i^*\mathcal{L}$ is a coherent sheaf of constant fiber dimension one, it is a line bundle by [Vak17, Exercise 13.7.K]. □

The *resistor network spectral transform* is the rational map

$$\rho_{G,v} : \mathcal{R}_N \to \mathcal{S}_N,$$

described on the torus chart $\mathcal{R}_G$ as follows:

1. C is the spectral curve.
2. Consider the section δ_{v_0} of $\bigoplus_{v \in V} \mathcal{O}_{(\mathbb{C}^\times)^2}$. Its image under the cokernel map $\bigoplus_{v \in V} \mathcal{O}_{(\mathbb{C}^\times)^2} \to \mathcal{L}$ is a section of $\mathcal{L}$. Restricting to C_0, we get a section of the line bundle $i^*\mathcal{L}$ on C_0. The divisor S is defined to be the divisor of this section.
3. ν is the parameterization of the points at infinity of C by zig-zag paths on G such that the coordinate of the point infinity associated with a zigzag path is determined by the weight of the zig-zag path. Precisely, let $[\alpha] = (i, j)$. Then $z^i w^j$ can be taken as a local coordinate on the line D_E, and the point $\nu(\alpha) \in D_E$ is defined by $z^i w^j = \frac{1}{wt(\alpha)}$.

By a rational map, we mean that the domain of the map $\rho_{G,v}$ is a Zariski-dense open subvariety of $\mathcal{R}_N$, that is, it is only defined for generic conductances.

If we take the image of the section δ_v of $\bigoplus_{v \in V} \mathcal{O}_{(\mathbb{C}^\times)^2}$ instead of δ_{v_0}, we denote the divisor we get by S_v, so $S = S_{v_0}$. The next proposition tells us how to compute the divisors S_v in practice. Let $Q(z, w)$ be the adjugate matrix of $\Delta(z, w)$.

Proposition 4.4. *The divisor S_v is the linear combination of points where the v-column of $Q(z, w)$ vanishes.*

Proof. Let s denote the image of the section δ_v in $\mathcal{L}$. The divisor S_v consists of the set of points in C_0 where s vanishes, that is, the points $(z, w) \in C_0$ where δ_v is in the image of $\Delta(z, w)$. We have $Q(z, w)\Delta(z, w) = \det \Delta(z, w) I = 0$. Now $\delta_v \in \operatorname{im} \Delta(z, w)$ means that there exists f such that $\Delta(z, w) f = 0$, which means that

$$Q(z, w)\Delta(z, w) f = Q(z, w)\delta_v = 0,$$

which means the v-column of $Q(z, w)$ vanishes. □

Remark 4.5. In fact, since corank $\Delta(z, w)$ is one, it suffices to consider the simultaneous vanishing of any two entries of the v-column of $Q(z, w)$.

4.4 The image of the spectral transform

For positive conductances, Kenyon has identified the open spectral curves that appear. Since we only need the first property, we do not give the definition of a simple Harnack curve here.

Theorem 4.6 ([Ken19]). *For the space $\mathcal{R}_N(\mathbb{R}_{>0})$ of positive-real-valued points of $\mathcal{R}_N$, we have*

1. *C_0 satisfies the three conditions of Proposition 2.7, and moreover the singular point $(1,1)$ is a node;*

2. *C_0 is a simple Harnack curve.*

Let $\sigma : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ denote the involution $(z, w) \mapsto (\frac{1}{z}, \frac{1}{w})$.

Lemma 4.7. *The point at infinity $\nu(\overline{\alpha}) = \sigma(\nu(\alpha))$.*

Proof. If α is a zig-zag path and $\overline{\alpha}$ is its conjugate, then $\nu(\alpha)$ and $\nu(\overline{\alpha})$ are defined by

$$z^i w^j(\nu(\alpha)) = \frac{1}{wt(\alpha)}, \quad z^{-i} w^{-j}(\nu(\overline{\alpha})) = \frac{1}{wt(\overline{\alpha})} = \frac{1}{wt(\alpha)}$$

respectively, where $(i, j) = [\alpha]$ and we have used (3). On the other hand, the point $\sigma(\nu(\alpha))$ also has coordinates

$$z^{-i} w^{-j}(\sigma(\nu(\alpha))) = \frac{1}{wt(\alpha)}.$$

□

Remark 4.8. More precisely, since N is centrally symmetric, σ extends to a toric morphism $\sigma : X_N \to X_N$, and the σ in $\sigma(\nu(\alpha))$ refers to this extension.

Let W be the subspace of curves C with Newton polygon N satisfying the following conditions:

1. $(1, 1) \in C$ and the point $(1, 1)$ is a node of C;

2. $\sigma\big|_C$ is an involution on C. For concision, we will denote $\sigma\big|_C$ by σ.

Let $\pi : \widehat{C} \to C$ denote the normalization of C. $\widehat{C}$ is a smooth curve such that $\pi^{-1}(1, 1)$ consists of two points q_1, q_2 that are glued together by π, while $\pi\big|_{\widehat{C}-\{q_1,q_2\}} : \widehat{C} - \{q_1, q_2\} \to C - \{(1, 1)\}$ is an isomorphism. Therefore the involution σ of C lifts to an involution $\widehat{\sigma}$ of $\widehat{C}$ such that q_1, q_2 are fixed points; we denote this involution also by σ. If S is a degree g effective divisor on $C - \{(1, 1)\}$, then $\widehat{S} := \pi^{-1}(S)$ is a degree g effective divisor in $\widehat{C}$.

Let $\mathcal{S}'_N$ be the moduli space of triples (C, S, ν) such that C is a curve in W, S is a degree g effective divisor on $C - \{(1, 1)\}$ satisfying

$$\widehat{S} + \widehat{\sigma}(\widehat{S}) - q_1 - q_2 \sim K_{\widehat{C}}, \tag{9}$$

where $\widehat{C}$ is the normalization of C, $K_{\widehat{C}}$ is the canonical divisor class of $\widehat{C}$, and ν is a parameterization of the points at infinity by zig-zag paths. The presence of the node $(1, 1)$ means that a generic curve in W has geometric genus g, one less than a generic curve with Newton polygon N.

We determine the image of the spectral transform. The proof is quite technical and undertaken in Section 5.

Theorem 4.9. *We have $\rho_{G,v}(\mathcal{R}_N) \subseteq \mathcal{S}'_N$.*

Proof.

1. For all positive-real-valued conductances, Theorem 4.6 tells us that $(1,1)$ is a node. Since nodes are characterized by non-vanishing of the Hessian, an open condition, $(1,1)$ is a node for all points in a Zariski open subset of $\mathcal{R}_N$. Along with Proposition 2.7, we get $C \in W$.
2. $\deg\ S = g$ is proved in Corollary 5.13.
3. $\widehat{S} + \widehat{\sigma}(\widehat{S}) - q_1 - q_2 = K_{\widehat{C}}$ is Corollary 5.12.

□

The condition (9) says that there exists a meromorphic 1-form on $\widehat{C}$ that has zeros at the $2g$ points $\widehat{S}+\widehat{\sigma}(\widehat{S})$ and poles at q_1, q_2. We write down this 1-form explicitly in Proposition 4.10 below. The proof is a technical computation of the zeros and poles of ω. Since the result is not used elsewhere in the paper, it may be skipped.

Proposition 4.10. *Let $R(z,w) = Q_{v_0,v_0}(z,w)$ be the minor of $\Delta(z,w)$ with the row and column corresponding to v_0 removed. The meromorphic 1-form*

$$\omega = \pi^* \left(\frac{R(z,w)dz}{zw\frac{\partial P(z,w)}{\partial w}} \right),$$

satisfies

$$\mathrm{div}_{\widehat{C}}\, \omega = \widehat{S} + \widehat{\sigma}(\widehat{S}) - q_1 - q_2.$$

Remark 4.11. The 1-form ω also appears in [BdTR17, Proposition 31]. The 1-form $\frac{R(z,w)dz}{zw\frac{\partial P(z,w)}{\partial w}}$ is defined on C_0, but since $C - C_0$ is a finite collection of isolated points, a meromorphic form on C_0 extends uniquely to C. ω is the pullback of this extension to $\widehat{C}$.

Proof of Proposition 4.10. For smooth $(z,w) \in C$, we have corank $\Delta(z,w) = 1$. Therefore, we can write $R(z,w) = U(z,w)V(z,w)^T$ for some $U(z,w) \in \ker\ \Delta(z,w), V(z,w) \in \mathrm{coker}\ \Delta(z,w)$. By definition, S is the set of points in C_0 where the component $V(z,w)\cdot\delta_{v_0}$ of $V(z,w)$ vanishes. We have $\ker\ \Delta(z,w) \cong \mathrm{coker}\ \Delta(z,w)^T = \mathrm{coker}\ \Delta(\frac{1}{z},\frac{1}{w})$, so $\sigma(S)$ are the points where the component $U(z,w)\cdot\delta_{v_0}$ vanishes. Since $R(z,w) = (U(z,w)\cdot\delta_{v_0})(V(z,w)\cdot\delta_{v_0})$, we have

$$\mathrm{div}_{C_0} R(z,w) = S + \sigma(S),$$

Since C has a node at $(1,1)$, $\frac{\partial P(z,w)}{\partial w}$ has a simple zero at $(1,1)$ and so ω has simple poles at q_1, q_2. Therefore, the divisor of ω on the complement of the points at infinity is $\widehat{S}+\widehat{\sigma}(\widehat{S}) - q_1 - q_2$, which has degree $2g-2$. It remains to identify the zeros and poles of ω at the points at infinity.

The order of vanishing of the 1-form

$$\omega_{ij} := \frac{z^{i-1}w^{j-1}dz}{\frac{\partial P(z,w)}{\partial w}}$$

at the point at infinity corresponding to the primitive integral edge E is given by the twice the signed area of the triangle formed by E and the point (i,j) minus one (where the area is positive for points (i,j) inside N). $R(z,w)$ is the partition function of OCRSFs on the graph G' obtained from G by removing the vertex v_0. By Corollary 3.3, the Newton polygon of $R(z,w)$ is strictly contained in N. Therefore, the order of vanishing of ω must be non-negative at all points at infinity, that is, ω has no poles at these points. The divisor of ω on the complement of the points at infinity has degree $2g-2$ (Corollary 5.13), which is the degree of $K_{\widehat{C}}$. Therefore, ω must have an equal number of zeros and poles at the points at infinity, and therefore ω also has no zeros at infinity. □

5 Holomorphic extension of the line bundle Laplacian to C

Recall that $i : C_0 \hookrightarrow (\mathbb{C}^\times)^2$ denotes the inclusion and $\pi : \widehat{C} \to C$ is the normalization map. Consider the commuting diagram:

$$\begin{array}{ccccc} \widehat{C}_0 & \xrightarrow{\pi} & C_0 & \xhookrightarrow{i} & (\mathbb{C}^\times)^2 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \widehat{C} & \xrightarrow{\pi} & C & \hookrightarrow & X_N \end{array}, \tag{10}$$

where $\widehat{C}_0 := \pi^{-1}(C_0)$ is $\widehat{C}$ minus the points at infinity. To prove Theorem 4.9, we need to use the theory of divisors and line bundles on smooth Riemann surfaces. As we mentioned at the beginning of Section 4, there were two obstructions that we have now addressed:

1. C_0 is not compact: We compactified C_0 to C;
2. C_0 (resp. C) has a node at $(1,1)$: We resolved the node using a normalization to get $\widehat{C}_0$ (resp. $\widehat{C}$).

However, the line bundle $i^*\mathcal{L}$ used to define the spectral transform is still defined on C_0, and we would like to have a line bundle on $\widehat{C}$. Pulling back along $\pi : \widehat{C}_0 \to C_0$, we get a line bundle $\pi^* i^* \mathcal{L}$ on $\widehat{C}_0$, but now there is no canonical way to extend a line bundle on $\widehat{C}_0$ to $\widehat{C}$, since we can have twists at the points of $\widehat{C} - \widehat{C}_0$. However, since our line bundle comes from the line bundle Laplacian, we will find a canonical extension by first extending the line bundle Laplacian holomorphically to $\widehat{C}$.

We pull back (8) using $\pi^* i^*$ and use the right exactness of the pullback to get the following exact sequence on $\widehat{C}_0$:

$$\bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}_0} \xrightarrow{\pi^* i^* \Delta} \bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}_0} \to \pi^* i^* \mathcal{L} \to 0. \tag{11}$$

Now, we need to extend the trivial vector bundle $\bigoplus_{v \in V} \mathcal{O}_{\widehat{C}_0}$ from $\widehat{C}_0$ to $\widehat{C}$. As is usual in algebraic geometry, we will often implicitly identify a line bundle, its invertible sheaf of holomorphic/regular sections, and the divisor of a meromorphic section. We recall the correspondences in A.1 and refer, for example, to [Mir95, Chapter XI] for details. Precisely, for each of the direct summands in $\bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}_0}$, we want an invertible sheaf $\mathcal{F}$ on $\widehat{C}$ such that $\mathcal{F}\big|_{\widehat{C}_0} = \mathcal{O}_{\widehat{C}_0}$. Every invertible sheaf on $\widehat{C}$ is of the form $\mathcal{F} = \mathcal{O}_{\widehat{C}}(D)$ for a divisor D in $\widehat{C}$, that is,

$$\mathcal{O}_{\widehat{C}}(D)(U) = \{t \in K(\widehat{C})^\times : \operatorname{div}\big|_U t + D\big|_U \geq 0\} \cup \{0\},$$

where $K(\widehat{C})^\times$ is the space of nonzero rational functions on $\widehat{C}$. Letting $U = \widehat{C}_0$, we see that $\mathcal{F}\big|_{\widehat{C}_0} = \mathcal{O}_{\widehat{C}_0}$ means $D\big|_U = 0$, so D is supported at the points at infinity of $\widehat{C}$. Therefore the extension of $\bigoplus_{v\in V(G)} \mathcal{O}_{\widehat{C}_0}$ is given by a divisor at infinity of $\widehat{C}$ for every vertex $v \in V(G)$. The divisors at infinity will be determined from the combinatorics of the resistor network via the discrete Abel map of Fock [Foc15].

5.1 The discrete Abel map

Recall that we denote the zig-zag path oriented opposite to α by $\overline{\alpha}$, and that $\widetilde{G}$ denotes the biperiodic resistor network in the plane, that is the preimage of G under the universal covering map $p : \mathbb{R}^2 \to \mathbb{T}$. Let $V(\widetilde{G})$ and $F(\widetilde{G})$ denote the set of vertices and faces of $\widetilde{G}$. Let $\mathbb{Z}^Z$ denote the group of $\mathbb{Z}$-linear combinations of zig-zag paths. Define $\widetilde{d} : V(\widetilde{G}) \cup F(\widetilde{G}) \to \mathbb{Z}^Z$ as follows:

Set $\widetilde{d}(v_0) = 0$ for some vertex v_0. For any vertex or face u, let $\widetilde{\gamma}$ be a path from v_0 to u in $\widetilde{G}$. Let

$$\widetilde{d}(u) = \sum_{\alpha\in Z} \sum_{\widetilde{\alpha}\in p^{-1}(\alpha)} (\alpha, \gamma)\alpha,$$

where $(\cdot,\cdot)$ is the intersection pairing in the plane and the second sum is over all lifts of $\widetilde{\alpha}$ of α to the plane. In other words, $\widetilde{d}(u)$ keeps track of zig-zag paths of $\widetilde{G}$ crossed by any path from v_0 to u. As such, $\widetilde{d}$ is defined on $\widetilde{G}$, but is not well defined on G. Let $u + (i, j)$ denote the translate of u in $\widetilde{G}$ by $i\gamma_z + j\gamma_w$. Then we have

$$\begin{aligned}\widetilde{d}(u + (i, j)) - \widetilde{d}(u) &= \sum_{\alpha\in Z} \sum_{\widetilde{\alpha}\in p^{-1}(\alpha)} (\alpha, i\gamma_z + j\gamma_w)\alpha \\ &= \sum_{\alpha\in Z} ([\alpha], (i, j))_{\mathbb{T}}\alpha,\end{aligned}$$

where $(\cdot,\cdot)_{\mathbb{T}}$ is the intersection pairing in $\mathbb{T}$:

$$\begin{aligned}(\cdot,\cdot)_{\mathbb{T}} : H_1(\mathbb{T}, \mathbb{Z}) \times H_1(\mathbb{T}, \mathbb{Z}) &\to \mathbb{Z} \\ ((a, b), (c, d))_{\mathbb{T}} &= ad - bc.\end{aligned}$$

Therefore, we define the inclusion

$$\begin{aligned}\rho : H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2 &\hookrightarrow \mathbb{Z}^Z \\ h &\mapsto \sum_{\alpha\in Z} ([\alpha], h)_{\mathbb{T}}\alpha.\end{aligned}$$

Then $\widetilde{d}$ is equivariant with respect to the $H_1(\mathbb{T}, \mathbb{Z})$-action, that is,

$$\widetilde{d}(u + h) = \widetilde{d}(u) + \rho(h),$$

for all $u \in V(\widetilde{G}) \cup F(\widetilde{G})$. Applying the parameterization ν of the points at infinity to $\mathbb{Z}^Z$, we think of $\mathbb{Z}^Z$ as divisors at infinity of $\widehat{C}$. To keep the notation concise, we will usually write α for the point at infinity $\nu(\alpha)$. Then the following proposition says that the image of the map ρ consists of divisors of the monomials $z^i w^j$ in $(\mathbb{C}^\times)^2$, and in particular $\nu \circ \rho(h)$ is a prinicpal divisor for every $h \in H_1(\mathbb{T}, \mathbb{Z})$.

Proposition 5.1. *We have $\nu \circ \rho(i, j) = \operatorname{div}_{\widehat{C}} z^i w^j$.*

Proof. Let E be an edge of N and let (a,b) denote the primitive integral vector normal to E, oriented towards the inside of N. By [CLS11, Proposition 4.1.1], the order of vanishing of z^iw^j at the line at infinity D_E of X_N is $ia+jb$. Restricting to $\widehat{C}$, we get that the order of vanishing of z^iw^j at a point at infinity $\alpha \in Z(E)$ is $ia+jb = ([\alpha],(i,j))_{\mathbb{T}}$. □

Following Fock [Foc15], we define the discrete Abel map $d : V(G) \cup F(G) \to \mathbb{Z}^Z$ as follows: Let v_0 be a vertex of G. Let R be a fundamental rectangle as in Figure 6. For each vertex $v \in V(G)$, let $\widetilde{v}$ denote the lift of v contained in R, and define $d(v)$ to be $\widetilde{d}(\widetilde{v})$.

Lemma 5.2. *For all edges $e : u \to v$ of G with pairs of oriented zig-zag paths $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ through e, we have*

$$d(v) - d(u) = -\alpha - \beta + \overline{\alpha} + \overline{\beta} - \operatorname{div}_{\widehat{C}} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}.$$

Proof. Let $\widetilde{u}$ and $\widetilde{v}$ be the lifts of u and v in R. Let $\widetilde{e}$ denote the lift of e that is incident to $\widetilde{u}$. Then the other end point of $\widetilde{e}$ is $\widetilde{v} + ((e,\gamma_z)_{\mathbb{T}}, (e,\gamma_w)_{\mathbb{T}})$. Therefore

$$\begin{aligned} d(v) - d(u) &= \widetilde{d}(\widetilde{v}) - \widetilde{d}(\widetilde{u}) \\ &= \widetilde{d}(\widetilde{v} + ((e,\gamma_z)_{\mathbb{T}}, (e,\gamma_w)_{\mathbb{T}})) - \operatorname{div}_{\widehat{C}} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}} - \widetilde{d}(\widetilde{u}) \\ &= -\alpha - \beta + \overline{\alpha} + \overline{\beta} - \operatorname{div}_{\widehat{C}} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}. \end{aligned}$$

□

Example 5.3. Let us compute the discrete Abel map for the network in Figure 2, with zig-zag paths labeled as in Figure 4. We have (with $v_0 := v_1$):

$$\begin{aligned} d(v_1) &= 0, \\ d(v_2) &= -\alpha - \beta + \overline{\alpha} + \overline{\beta}. \end{aligned}$$

We also compute $\rho : H_1(\mathbb{T},\mathbb{Z}) \hookrightarrow \mathbb{Z}^Z$:

$$\begin{aligned} (1,0) \quad &\mapsto \quad -2\alpha - 2\beta + 2\overline{\alpha} + 2\overline{\beta} \\ (0,1) \quad &\mapsto \quad \alpha - \beta - \overline{\alpha} + \overline{\beta}. \end{aligned} \tag{12}$$

5.2 Construction of the extension

Define the line bundles

$$\begin{aligned} \mathcal{F}_v &= \mathcal{O}_{\widehat{C}}\left(d(v) - \sum_{\alpha \in Z : v \in \alpha} \alpha\right), \\ \mathcal{G}_v &= \bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}}(d(v)), \end{aligned}$$

and the vector bundles

$$\mathcal{F} = \bigoplus_{v \in V(G)} \mathcal{F}_v, \quad \mathcal{G} = \bigoplus_{v \in V(G)} \mathcal{G}_v.$$

The sum in the definition of $\mathcal{F}_v$ is over all zig-zag paths that contain the vertex v. The meromorphic function Δ_{vu} defines a meromorphic section $\widehat{\Delta}_{vu}$ of the line bundle $\mathcal{H}om(\mathcal{F}_u, \mathcal{G}_v)$ (see A.1.5). Therefore, we get an extension of (11) to a meromorphic map of vector bundles on $\widehat{C}$

$$\mathcal{F} \xrightarrow{\widehat{\Delta}} \mathcal{G}, \tag{13}$$

which is a $V(G) \times V(G)$ matrix with entries $\widehat{\Delta}_{vu}$.

Remark 5.4. More precisely, $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{u \in V(G)} \bigoplus_{v \in V(G)} \mathcal{H}om(\mathcal{F}_u, \mathcal{G}_v), \widehat{\Delta} \mapsto (\widehat{\Delta}_{vu})_{u,v \in V(G)}$.

Theorem 5.5. *The map $\widehat{\Delta}$ is holomorphic, that is, for every $u, v \in V(G)$, $\widehat{\Delta}_{vu}$ is a holomorphic section of $\mathcal{H}om(\mathcal{F}_u, \mathcal{G}_v)$.*

Proof. We need to show that for each $v, w \in V$, the component $\widehat{\Delta}_{vu}$ is a holomorphic section of $\mathcal{H}om(\mathcal{F}_u, \mathcal{G}_v) \cong \mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha\right)$, that is $\operatorname{div} \Delta_{vu} + d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha \geq 0$. By definition of the line bundle Laplacian $\Delta(z, w)$, we have

$$\Delta_{vu}(z,w) = \begin{cases} \sum_{e: v' \to v: v' \neq v} c(e) + \sum_{e: v \to v} c(e)(1 - z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}) & \text{if } v = u; \\ -\sum_{e: u \to v} c(e) z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}} & \text{otherwise.} \end{cases}$$

When $v \neq u$, recall that for each edge $e : u \to v$ we have by Lemma 5.2 that

$$d(v) - d(u) = -\beta - \delta + \overline{\beta} + \overline{\delta} - \operatorname{div} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}},$$

where $\beta, \delta, \overline{\beta}, \overline{\delta}$ are the oriented zig-zag paths through e, with $\beta, \overline{\beta}$ and $\delta, \overline{\delta}$ the oppositely oriented pairs. From this we get

$$\operatorname{div} z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}} + d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha = \sum_{\alpha \in Z: v \in \alpha, \alpha \neq \beta, \delta, \overline{\beta}, \overline{\delta}} \alpha \geq 0,$$

so each $z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}$ is a holomorphic section of $\mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha\right)$. Since Δ_{vu} is a linear combination of these, the same holds for it as well.

When $v = u$, Δ_{vu} is a sum of constant terms in z, w and terms that involve $z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}}$ as in the case $u \neq v$. $d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha = \sum_{\alpha \in Z: v \in \alpha} \alpha \geq 0$ implies that the constant terms are also holomorphic sections of $\mathcal{O}_{\widehat{C}}\left(d(v) - d(u) + \sum_{\alpha \in Z: v \in \alpha} \alpha\right)$. $\square$

Let $\widehat{\mathcal{L}} := \operatorname{coker} \widehat{\Delta}$ and $\widehat{\mathcal{M}} := \ker \widehat{\Delta}$. The following commuting diagram shows how everything fits together:

$$\begin{array}{ccccccc} \mathcal{F} & \xrightarrow{\widehat{\Delta}} & \mathcal{G} & \longrightarrow & \widehat{\mathcal{L}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}_0} & \xrightarrow{\pi^* i^* \Delta} & \bigoplus_{v \in V(G)} \mathcal{O}_{\widehat{C}_0} & \longrightarrow & \pi^* i^* \mathcal{L} & \longrightarrow & 0 \end{array}$$

The downward maps are all restriction from $\widehat{C}$ to $\widehat{C}_0$.

5.3 Proof of Theorem 4.9

By Proposition 4.4, we know that the divisor S is given by the simultaneous vanshing of the v_0-column of the adjugate matrix $Q(z,w)$ of $\Delta(z,w)$. We need the following elementary linear algebra fact.

Proposition 5.6. *If A is an $n\times n$ matrix of rank $n-1$ and U,V are the matrices of the kernel and cokernel maps:*

$$0\to\mathbb{C}\xrightarrow{U}\mathbb{C}^n\xrightarrow{A}\mathbb{C}^n\xrightarrow{V}\mathbb{C}\to 0.$$

Then adj $A=UV^T$.

Since $\Delta(z,w)$ has corank one, by Proposition 5.6, we get that $Q(z,w)$ is of the form $U(z,w)V(z,w)^T$ where $U(z,w)$ and $V(z,w)$ are the matrices of the kernel and cokernel maps. Therefore, $Q_{v_0,v_0}(z,w)=U_{v_0}(z,w)V_{v_0}(z,w)$. By definition, the divisor S is the set of points where $U_{v_0}(z,w)=0$. Taking transpose and using the symmetry $\Delta(z,w)^T=\Delta(\frac{1}{z},\frac{1}{w})$, the set of points where $V_{v_0}(z,w)=0$ is $\sigma(S)$. Therefore, $Q_{v_0,v_0}(z,w)=0$ at the points of $S+\sigma(S)$, which is the divisor in (9).

Our strategy is to study the adjugate matrix $\widehat{Q}$ of $\widehat{\Delta}$. Since $\widehat{\Delta}$ is an extension of $\pi^*i^*\Delta$, we have

$$\operatorname{div}\big|_{\widehat{C}_0}\widehat{Q}_{v_0,v_0}=\widehat{S}+\widehat{\sigma}(\widehat{S}).$$

By analyzing the behaviour of $\widehat{Q}_{v_0,v_0}$ at the points at infinity of $\widehat{C}$, we can compute its divisor explicitly (Corollary 5.11). On the other hand, $\widehat{Q}_{v_0,v_0}$ is an exterior power of (13), which will give us an expression for its divisor class (Corollary 5.8). Comparing the two will give us (9).

A *toric divisor* on the toric surface X_N is a formal linear combination of the lines at infinity D_E of X_N. There are two special toric divisors that will play a role in our computations below.

1. The polygon N determines a divisor D_N as follows [CLS11, (4.2.7)]: Let u_E denote the primitive integral vector normal to E, and let $a_E\in\mathbb{Z}$ denote the distance from the origin to E along the direction of u_E (precisely, a_E is defined such that E is on the line $\langle(i,j),u_E\rangle=-a_E$, where $\langle\cdot,\cdot\rangle$ is the standard inner product).
2. The *canonical divisor* of X_N is $K_{X_N}=-\sum_{E\text{ edge of }N}D_E$ [CLS11, Theorem 8.2.3].

Taking the determinant of (13), we get a holomorphic map $\det\widehat{\Delta}:\bigwedge_{v\in V(G)}\mathcal{F}_v\to\bigwedge_{v\in V(G)}\mathcal{G}_v$, that is $\det\widehat{\Delta}$ is a holomorphic section of

$$\begin{aligned}\mathcal{H}om\left(\bigwedge_{v\in V(G)}\mathcal{F}_v,\bigwedge_{v\in V(G)}\mathcal{G}_v\right)&\cong\mathcal{H}om\left(\bigwedge_{v\in V(G)}\mathcal{O}_{\widehat{C}}\left(d(v)-\sum_{\alpha\in Z:v\in\alpha}\alpha\right),\bigwedge_{v\in V(G)}\mathcal{O}_{\widehat{C}}(d(v))\right)\\&\cong\mathcal{O}_{\widehat{C}}\left(\sum_{v\in V(G)}(d(v))-\sum_{v\in V(G)}\left(d(v)-\sum_{\alpha\in Z:v\in\alpha}\alpha\right)\right),\end{aligned}\tag{14}$$

where we have used Proposition A.4 to get the second isomorphism. We wish to identify this line bundle. The restriction $D_N\big|_C$ of D_N to C is a divisor at infinity of C. Therefore, $\pi^{-1}(D_N\big|_C)$ is a divisor at infinity of $\widehat{C}$.

Proposition 5.7. $\det\widehat{\Delta}$ *is a holomorphic section of* $\mathcal{O}_{\widehat{C}}(\pi^{-1}(D_N\big|_C))$.

Proof. Recall that $D_N = \sum_{E \text{ edge of } N} a_E D_E$, where $a_E \in \mathbb{Z}$ is the distance from the origin to E along the primitive normal vector to E. Therefore,

$$\begin{aligned} D_N|_{\widehat{C}} &= \sum_{\text{edges } E \text{ of } N} a_E D_E \cap \widehat{C} \\ &= \sum_{\text{edges } E \text{ of } N} \sum_{\alpha \in Z : [\alpha] \in E} a_E \alpha, \end{aligned}$$

where the inner sum is over all zig-zag paths α such that $[\alpha]$ is a primitive vector in E. By (14), we need to show that

$$\sum_{v \in V(G)} (d(v)) - \sum_{v \in V(G)} \left(d(v) - \sum_{\alpha \in Z : v \in \alpha} \alpha \right) = \pi^{-1}(D_N|_{\widehat{C}}).$$

Let β be a zig-zag path, let (i_1, i_2) be a vertex of N incident to the edge of N corresponding to β and let F be the corresponding extremal OCRSF. From our description of extremal OCRSFs (Theorem 3.2), we know that for each vertex $u \in V(G)$, there is a unique outgoing edge e_u and that if β contains u, then $e_u \in \beta$. We pair vertices of G using e_v to rewrite the sum as

$$\sum_{e : u \to v \in F} d(v) - d(u) + \sum_{\alpha \in Z : u \in \alpha} \alpha.$$

Now we observe that if $e \in \beta$, then β appears twice in the summand with opposite signs and if $e \notin \beta$, then β does not appear in the summand, modulo contributions from the edges of F intersecting γ_z, γ_w. This latter contribution is given by

$$\begin{aligned} -\sum_{e \in F} \text{div } z^{(e,\gamma_z)_{\mathbb{T}}} w^{(e,\gamma_w)_{\mathbb{T}}} &= -z^{(i_1,\gamma_z)_{\mathbb{T}}} w^{(i_2,\gamma_w)_{\mathbb{T}}} \\ &= a_E. \end{aligned}$$

□

Let $\widehat{Q}$ denote the adjugate matrix of $\widehat{\Delta}$. Then $\widehat{Q}_{vu}$ is a holomorphic map $\widehat{Q} : \bigwedge_{w \in V(G) - \{u\}} \mathcal{F}_w \to \bigwedge_{w \in V(G) - \{v\}} \mathcal{G}_w$, that is a holomorphic section of

$$\begin{aligned} &\mathcal{H}om \left(\bigwedge_{w \in V(G) - \{u\}} \mathcal{F}_w, \bigwedge_{w \in V(G) - \{v\}} \mathcal{G}_w \right) \\ &\cong \mathcal{O}_{\widehat{C}} \left(\sum_{w \in V(G) - \{u\}} (d(w)) - \sum_{w \in V(G) - \{v\}} \left(d(w) - \sum_{\alpha \in Z : w \in \alpha} \alpha \right) \right) \\ &\cong \mathcal{O}_{\widehat{C}} \left(D_N|_{\widehat{C}} - d(v) + d(u) - \sum_{\alpha \in Z : u \in \alpha} \alpha \right), \end{aligned}$$

where the first isomorphism is using Proposition A.4, and the second isomorphism is from Proposition 5.7.

Corollary 5.8. *$\widehat{Q}_{vu}$ is a holomorphic section of $\mathcal{O}_{\widehat{C}}\left(\pi^{-1}(D_N\big|_{\widehat{C}}) - d(v) + d(u) - \sum_{\alpha\in Z:u\in\alpha}\alpha\right)$.*

Example 5.9. Going back to our example in Figure 2, we check that the Laplacian that we computed in (7) extends to a morphism of vector bundles (computed using the discrete Abel map in Example 5.3):

$$\mathcal{O}(-\alpha-\beta-\overline{\alpha}-\overline{\beta})\oplus\mathcal{O}(-2\alpha-2\beta)\to\mathcal{O}\oplus\mathcal{O}(-\alpha-\beta+\overline{\alpha}+\overline{\beta}). \tag{15}$$

We have $\Delta(z,w)_{v_1v_2} = -a - bz$, which we wish to show corresponds to a regular section of $\mathcal{O}(2\alpha + 2\beta)$. We check the following.

$$\begin{aligned}\text{div }\ a+2\alpha+2\beta &= 0+2\alpha+2\beta\geq 0,\\ \text{div }\ bz+2\alpha+2\beta &= (-2\alpha-2\beta+2\overline{\alpha}+2\overline{\beta})+2\alpha+2\beta\\ &= 2\overline{\alpha}+2\overline{\beta}\geq 0,\end{aligned}$$

where we have used div $z = -2\alpha - 2\beta + \overline{2}\alpha + \overline{2}\beta$ from (12) and Proposition 5.1. The other entries of $\Delta(z,w)$ can be checked in the same way.

Consider the edge E corresponding to the zig-zag path β in Figure 4. The primitive normal vector is $u_E = (-2,-1)$. Then a_E is defined as the intercept of the line containing E:

$$E\subset\{(i,j)\in\mathbb{R}^2 : -2i-j+a_E=0\}.$$

Since $(1,0)\in E$, we get $a_E = 2$. Similarly computing the intercepts for the other zig-zag paths, we get

$$D_N = 2\alpha+2\beta+2\overline{\alpha}+2\overline{\beta}.$$

On the other hand, taking the determinant of (15), we compute the determinant line bundle using Proposition A.4 to be

$$\begin{aligned}&\mathcal{H}om\left(O(-\alpha-\beta-\overline{\alpha}-\overline{\beta})\wedge\mathcal{O}(-2\alpha-2\beta),\mathcal{O}\wedge\mathcal{O}(-\alpha-\beta+\overline{\alpha}+\overline{\beta})\right)\\ &\cong\mathcal{O}(2\alpha+2\beta+2\overline{\alpha}+2\overline{\beta}),\end{aligned}$$

verifying the conclusion of Lemma 5.7.

Corollary 5.8 tells us the linear equivalence class of $\text{div}_{\widehat{C}}\,\widehat{Q}$. Next we perform a careful analysis of the behaviour of $\widehat{Q}_{uv}$ at the points at infinity of $\widehat{C}$ to determine its divisor. Consider the exact sequence

$$0\to\widehat{\mathcal{M}}\xrightarrow{\begin{bmatrix}|\\ s_v\\ |\end{bmatrix}}\mathcal{F}\xrightarrow{\widehat{\Delta}}\mathcal{G}\xrightarrow{\begin{bmatrix}-t_v-\end{bmatrix}}\widehat{\mathcal{L}}\to 0,$$

where s_v is the holomorphic section of $\mathcal{H}om(\widehat{\mathcal{M}},\mathcal{F}_v)$ given by the v-entry of the kernel map and t_v is the holomorphic section of $\mathcal{H}om(\mathcal{G}_v,\widehat{\mathcal{L}})$ given by the v-entry of the cokernel map. By Proposition 5.6, we have $\widehat{Q}_{uv} = s_u\cdot t_v$, so we compute the divisors of s_v and t_v.

Recall that S_v is the effective divisor given by the vanishing of the v-column of $Q(z,w)$ and $\widehat{S}_v = \pi^{-1}(S_v)$.

Proposition 5.10. *We have*

$$\operatorname{div}_{\widehat{C}} s_v = \widehat{\sigma}(\widehat{S}_v) + \sum_{\alpha \in Z : u \notin \alpha} \alpha;$$
$$\operatorname{div}_{\widehat{C}} t_v = \widehat{S}_v.$$

Proof. Let α be a zig-zag path. Let U be a neighbourhood of the associated point at infinity α of $\widehat{C}$ that does not contain any other points at infinity. Let x be a local parameter with a simple zero at α. We trivialize the line bundles in (13) as follows:

$$\mathcal{O}(-k\alpha)(U) \xrightarrow{\cong} \mathcal{O}(U)$$
$$f \mapsto x^{-k} f$$

Let $z = ax^m + O(x^{m+1})$ and $w = bx^n + O(x^{n+1})$ be the expansions in the local coordinate x, where $O(x^l)$ denotes a function vanishing to order at least l. Let us order the vertices so that the vertices in the zig-zag path α appear first. Then near the point α of $\widehat{C}$, $\widehat{\Delta}$ has the following block form:

$$\widehat{\Delta} = \begin{pmatrix} \Delta_1 & B \\ xA & \Delta_2 \end{pmatrix} + O(x),$$

where Δ_1 is the restriction of $\widehat{\Delta}$ to the zig-zag path α and Δ_2 is the restriction to the rest of G, and where z and w are replaced with a and b respectively. Since we are at α, Δ_1 is singular. For smooth $\widehat{C}$, $\dim \ker\ \Delta_1 = 1$ and Δ_2 is invertible.

Let $g \in \ker\ \Delta_1^*$. Then we have

$$\ker\ \widehat{\Delta}^T = (g, -(\Delta_2^T)^{-1} B^T g) + O(x).$$

Since generically none of the entries in $\ker\ \widehat{\Delta}^T$ is 0, and since these entries are the cofactors of $\widehat{\Delta}$, we see that t_v has no poles or zeros at α. Since α was arbitrary, t_v has no zeros or poles at infinity.

Now let $g \in \ker\ \Delta_1$. We have

$$\ker\ \widehat{\Delta} = (g, -x\Delta_2^{-1} A g) + O(x),$$

from which we see that s_u has a simple zero at α if $u \notin \alpha$ and no zeros or poles at α if $u \in \alpha$. ☐

Now, since $\widehat{Q}_{uv} = s_u \cdot t_v$, we get:

Corollary 5.11. *The divisor of* $\widehat{Q}_{vu}$ *is* $\operatorname{div}_{\widehat{C}} \widehat{Q}_{vu} = \widehat{S}_v + \widehat{\sigma}(\widehat{S}_u) + \sum_{\alpha \in Z : u \notin \alpha} \alpha$.

Corollary 5.12. *We have* $\widehat{S} + \widehat{\sigma}(\widehat{S}) - q_1 - q_2 \sim K_{\widehat{C}}$.

Proof. From Corollary 5.8 and Corollary 5.11, we get that

$$\operatorname{div}_{\widehat{C}} \widehat{Q}_{v_0 v_0} = \widehat{S} + \widehat{\sigma}(\widehat{S}) + \sum_{\alpha \in Z : u \notin \alpha} \alpha \sim \pi^{-1}(D_N\big|_C) - d(v) + d(u) - \sum_{\alpha \in Z : u \in \alpha} \alpha,$$

rearranging which we get

$$\widehat{S} + \widehat{\sigma}(\widehat{S}) - q_1 - q_2 \sim \pi^{-1}(D_N\big|_C) - \sum_{\alpha \in Z} \alpha - q_1 - q_2.$$

Recall that the canonical divisor of the toric variety X_N is given by $K_{X_N} = -\sum_{E \text{ edge of } N} D_E$. By the adjunction formula for nodal curves [ACGH85, Appendix A.8], we have

$$\begin{aligned} K_{\widehat{C}} &= \pi^{-1}\left(-\sum_E D_E + D_N\right)\Big|_C - q_1 - q_2 \\ &= \pi^{-1}(D_N\big|_C) - \sum_{\alpha \in Z} \alpha - q_1 - q_2. \end{aligned}$$

□

Corollary 5.13. deg $S_v = g$ *for all* $v \in V(G)$.

Proof. We take degrees on both sides of $\widehat{S}_v + \widehat{\sigma}(\widehat{S}_v) - q_1 - q_2 \sim K_{\widehat{C}}$, and use deg $K_{\widehat{C}} = 2g - 2$. □

6 Inverse spectral transform

In this section, we construct the inverse spectral transform using theta functions on the Prym variety of $(\widehat{C}, \widehat{\sigma})$.

6.1 Curves and their Jacobians

In this section, we review some results about the Jacobian variety. For further details, we refer to the books [Fay73, Mum07a, Mum07b].

Let C be a compact Riemann surface/smooth curve of genus g. Let $(A_i, B_i)_{i=1}^g$ be a canonical basis for $H_1(C, \mathbb{Z})$, so that

$$A_i \cdot A_j = 0, \quad B_i \cdot B_j = 0, \quad A_i \cdot B_j = \delta_{ij},$$

where $\cdot$ is the intersection pairing on C. Let K_C denote the canonical divisor of C (i.e. the divisor class of 1-forms) and let $(\omega_i)_{i=1}^g$ be the basis of the vector space $H^1(C, K_C)$ of holomorphic 1-forms, dual to $(A_i, B_i)_{i=1}^g$:

$$\int_{A_i} \omega_j = \delta_{ij}.$$

We have the period map

$$\begin{aligned} \text{per} : H_1(C, \mathbb{Z}) &\hookrightarrow \mathbb{C}^g \\ \gamma &\mapsto \left(\int_\gamma \omega_i\right)_{i=1}^g, \end{aligned}$$

identifying $H_1(C, \mathbb{Z})$ with a lattice in $\mathbb{C}^g$, called the *period lattice*. The *Jacobian variety* of C is defined as

$$J(C) := \mathbb{C}^g / \text{per}(H_1(C, \mathbb{Z})).$$

If q is a base point, we define the *Abel map*:

$$\begin{aligned} I : C &\to J(C) \\ x &\mapsto \left(\int_q^x \omega_i\right)_{i=1}^g \text{ modulo per}(H_1(C, \mathbb{Z})). \end{aligned}$$

The *d-fold symmetric product* of C is defined as $C^{(d)} := C^d/S_d$, the quotient of C^d by the natural action of the symmetric group or equivalently, the space of degree d effective divisors on C. The Abel map naturally extends to $C^{(d)}$:

$$I : C^{(d)} \to J(C)$$
$$\sum_{i=1}^{d} x_i \mapsto \sum_{i=1}^{d}(I(x_i)).$$

For a divisor D, the *complete linear system* $|D|$ is the space of divisors linearly equivalent to D:

$$|D| := \{E \in C^{(d)} : E \sim D\}.$$

Equivalently $|D|$ is the space of holomorphic sections of $\mathcal{O}_C(D)$ modulo multiplication by a nonzero complex number

$$|D| = (H^0(C, \mathcal{O}_C(D)) - 0)/\mathbb{C}^\times = \mathbb{P}H^0(C, \mathcal{O}_C(D)),$$

because if $E \sim D$ is a degree d effective divisor, then $E - D \sim 0$ is a principal divisor, so $E - D = \operatorname{div}_C t$ for a meromorphic function t on C. The meromorphic function t gives a meromorphic section $\widetilde{t}$ of $\mathcal{O}_C(D)$ (see A.1.5) with divisor

$$\operatorname{div}_C \widetilde{t} = \operatorname{div}_C t + D = E.$$

Theorem 6.1 (Abel's theorem)**.** *Two effective divisors D_1 and D_2 of degree d on a smooth curve C are linearly equivalent if and only if $I(D_1) = I(D_2)$. Equivalently, the fibers of the Abel map are complete linear systems:*

$$I^{-1}(I(D)) = |D|.$$

Theorem 6.2 (Jacobi inversion theorem)**.** *Let C be a smooth curve of genus g. The Abel map $I : C^{(g)} \to J(C)$ is surjective and birational. Therefore, for a generic degree g effective divisor D, the complete linear system $I^{-1}(I(D)) = |D|$ is a point.*

Corollary 6.3. *If D is a generic degree g effective divisor, then the line bundle $\mathcal{O}_C(D)$ has a unique holomorphic section modulo multiplication by a nonzero complex number.*

6.2 The prime form

We follow [Mum07b, IIIb §1] and [Fay73, II]. Morally, the prime form should be a holomorphic function $E : C \times C \to \mathbb{C}$ on a compact Riemann surface C such that $E(x, y) = 0$ if and only if $x = y$. However, such a function cannot exist on a compact Riemann surface because a holomorphic function on C must have an equal number of zeros and poles, but it does exist as a holomorphic section of a line bundle on $C \times C$. Let $\Delta \subset C \times C$ be the diagonal (we only use this notation in this section, so there should be no confusion with the Laplacian).

Theorem 6.4. *There exists a unique holomorphic section E of $\mathcal{O}_{C\times C}(\Delta)$, called the prime form, such that*

1. *$E(x, y) = 0$ if and only if $x = y$.*
2. *E has a first order zero along Δ.*

3. $E(x,y) = -E(y,x)$.

4. $E(x,y)$ *vanishes like* $x-y$ *for* x,y *near* Δ.

For a divisor $D = \sum_i a_i - \sum_j b_j$ on $\widehat{C}$, we define

$$E_D(x) := \frac{\prod_i E(x,a_i)}{\prod_j E(x,b_j)}. \tag{16}$$

It is a section of the line bundle $\mathcal{O}_{\widehat{C}}(D)$ with $\operatorname{div}_{\widehat{C}} E_D(x) = D$. Let $D = \sum_i n_i x_i$ be a divisor of degree 0. There is a unique differential 1-form denoted ω_D on C with

1. zero A-periods, i.e., $\int_{A_i} \omega_D = 0$ for all $i = 1, \ldots, g$;
2. simple poles with residue n_i at p_i.

Then, $\omega_D(x) = d\log_x E_D(x)$ [Fay73, (21)], integrating which we get

$$e^{\int_q^x \omega_D} = \frac{E_D(x)}{E_D(q)}. \tag{17}$$

6.3 Prym varieties

For further background on the material collected here, see [Fay73, Fay89, Taĭ97].

Let $\widehat{C}$ be a smooth curve of genus g with a holomorphic involution $\sigma : \widehat{C} \to \widehat{C}$ with two fixed points q_1, q_2. Let $\overline{x} := \sigma(x)$ denote the conjugate point of $x \in \widehat{C}$. Let $p : \widehat{C} \to \widehat{C}/\sigma$ denote the ramified double cover, with branch points at q_1, q_2. We can choose a canonical homology basis for $H_1(\widehat{C}, \mathbb{Z})$

$$A_1, B_1, \quad A_2, B_2, \quad \ldots \quad A_g, B_g,$$

such that $(p_*A_i, p_*B_i)_{i=1}^{\frac{g}{2}}$ is a basis for $H_1(\widehat{C}/\sigma, \mathbb{Z})$, and such that

$$\sigma_* A_i + A_{i+\frac{g}{2}} = \sigma_* B_i + B_{i+\frac{g}{2}} = 0, \quad 1 \le k \le \frac{g}{2}. \tag{18}$$

Let $(u_1, \ldots, u_g)$ denote the basis of holomorphic differentials forms (i.e. basis of $H^0(\widehat{C}, K_{\widehat{C}})$) on $\widehat{C}$ dual to $(A_i, B_i)_{i=1}^g$, so that

$$\int_{A_i} u_j = \delta_{ij}.$$

Then for $1 \le j \le \frac{g}{2}$, we have

$$\sigma^* u_j + u_{j+\frac{g}{2}} = 0, \tag{19}$$

because

$$\begin{aligned}
\int_{A_i} (-\sigma^* u_j) &= -\int_{\sigma_* A_i} \sigma^* \sigma^* u_j \\
&= -\int_{-A_{i-\frac{g}{2}}} u_j \qquad \text{(using (18) and } \sigma^2 = \mathrm{id}\text{, so } (\sigma^*)^2 = \mathrm{id}) \\
&= \delta_{i, j+\frac{g}{2}},
\end{aligned}$$

which is the property that characterizes $u_{j+\frac{g}{2}}$.

A holomorphic differential form ω on $\widehat{C}$ is called a *Prym differential* if $\sigma^*\omega = -\omega$ (i.e. Prym differentials form the (-1)-eigenspace of $\sigma^* : H^0(\widehat{C}, K_{\widehat{C}}) \to H^0(\widehat{C}, K_{\widehat{C}})$). For $1 \le j \le \frac{g}{2}$, $\omega_j := u_j + u_{j+\frac{g}{2}}$ is a Prym differential because

$$\begin{aligned}\sigma^*\omega_j + \omega_j &= \sigma^* u_j + u_j + \sigma^* u_{j+\frac{g}{2}} + u_{j+\frac{g}{2}} \\ &= 0\end{aligned}$$

using (19). Moreover, it follows from (19) that $(\omega_j)_{j=1}^{\frac{g}{2}}$ is a basis for the vector space of Prym differentials on $\widehat{C}$. Let Π be the matrix of periods of the Prym differentials around the B-cycles of $\widehat{C}$:

$$\Pi_{jk} = \int_{B_k} \omega_j, \quad 1 \le j, k \le \frac{g}{2}.$$

The *Prym variety* $\mathrm{Pr}(\widehat{C}, \sigma)$ is defined to be $\mathbb{C}^{\frac{g}{2}}/(\mathbb{Z}^{\frac{g}{2}} + \Pi\mathbb{Z}^{\frac{g}{2}})$.

Let $J(\widehat{C})$ denote the Jacobian of $\widehat{C}$, and let $I : \widehat{C} \to J(\widehat{C})$ be the Abel map with base-point $q_0 \in \widehat{C}$. The involution σ induces an involution $\sigma_* : J(\widehat{C}) \to J(\widehat{C})$: Given $\zeta \in J(\widehat{C})$, let $D \in \widehat{C}^{(g)}$ (which exists by Theorem 6.2) such that $I(D) = \zeta$, and define $\sigma_*(\zeta) := I(\sigma(D))$. In coordinates, the induced map σ_* is given by

$$(z_1, \dots, z_g) \mapsto (-z_{\frac{g}{2}+1}, \dots, -z_g, -z_1, \dots, -z_{\frac{g}{2}}).$$

The Prym variety is embedded in the Jacobian via $\phi : \mathrm{Pr}(\widehat{C}, \sigma) \hookrightarrow J(\widehat{C})$:

$$(z_1, \dots, z_{\frac{g}{2}}) \mapsto (z_1, \dots, z_{\frac{g}{2}}, z_1, \dots, z_{\frac{g}{2}}).$$

We also have the projection $\pi_1 : J(\widehat{C}) \to \mathrm{Pr}(\widehat{C}, \sigma)$ given by

$$\pi_1(z_1, \dots, z_g) = (z_1 + z_{\frac{g}{2}+1}, \dots, z_{\frac{g}{2}} + z_g).$$

For $\zeta \in \phi(\mathrm{Pr}(\widehat{C}, \sigma))$, we have

$$\zeta = \frac{1}{2}\phi \circ \pi_1(\zeta). \tag{20}$$

Therefore, if $\phi(e) = \zeta$, then we can recover e as

$$e = \frac{1}{2}\pi_1(\zeta). \tag{21}$$

Define the *Abel-Prym map* with base-point q_1:

$$\begin{aligned}I_P &: \widehat{C} \to \mathrm{Pr}(\widehat{C}, \sigma) \\ &x \mapsto \left(\int_{q_1}^{x} \omega_1, \dots, \int_{q_1}^{x} \omega_{\frac{g}{2}}\right) \text{ modulo } \mathbb{Z}^{\frac{g}{2}} + \Pi\mathbb{Z}^{\frac{g}{2}}, \text{ for } x \in \widehat{C},\end{aligned}$$

so that we have $I_P = \pi_1 \circ I$ and $I_P(\sigma(x)) = -I_P(x)$.

The *Prym theta function* $\eta(z)$ is defined by

$$\eta(z) = \sum_{m \in \mathbb{Z}^{\frac{g}{2}}} e^{2\pi i m^T z + \pi i m^T \Pi m}, \quad z \in \mathbb{C}^{\frac{g}{2}}.$$

While $\eta(z)$ is called a function, it is only quasiperiodic under translations by $\mathbb{Z}^{\frac{g}{2}} + \Pi\mathbb{Z}^{\frac{g}{2}}$, and therefore, is actually a holomorphic section of a line bundle on $\mathrm{Pr}(\widehat{C}, \sigma)$. The key property of the Prym theta function is the following Prym analog of Riemann's theorem for the Jacobian.

Theorem 6.5 ([Fay73, Corollary 5.6])**.** *If $e \in Pr(\widehat{C}, \sigma)$, then either:*

1. *$\eta(I_P(x) - e) = 0$ for all $x \in \widehat{C}$, or*
2. *(generic case) $\mathrm{div}_{\widehat{C}}\eta(I_P(x) - e) = S$ is a degree g effective divisor satisfying*

$$\phi(e) = I(S) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}}) \in J(\widehat{C}),$$

and

$$S + \sigma(S) - q_1 - q_2 \sim K_{\widehat{C}},$$

where $K_{\widehat{C}}$ is the canonical divisor of $\widehat{C}$.

Proposition 6.6. *If $S \in \widehat{C}^{(g)}$ such that*

$$S + \sigma(S) - q_1 - q_2 \sim K_{\widehat{C}},$$

then

$$I(S) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}}) \in \phi(Pr(\widehat{C}, \sigma)).$$

Therefore, $I(S)$ is contained in a translate of the Prym variety inside the Jacobian $J(\widehat{C})$.

Proof. From the definition of the embedding ϕ, we have

$$\phi(\mathrm{Pr}(\widehat{C}, \sigma)) = \{\zeta \in J(\widehat{C}) : \sigma_*\zeta + \zeta = 0\}$$

We check that

$$\begin{aligned}
&I(S) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}}) + \sigma_*\left(I(S) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})\right)\\
&= I(S + \sigma(S)) - I(q_1) - I(q_2) - I(K_{\widehat{C}})\\
&= 0,
\end{aligned}$$

where we have used $\sigma_* I(K_{\widehat{C}}) = I(K_{\widehat{C}})$ (if $\mathrm{div}\,\omega = K_{\widehat{C}}$, then $\mathrm{div}\,\sigma^*\omega = \sigma_* K_{\widehat{C}}$, and then use Theorem 6.1). ☐

The following theorem is the Prym version of Fay's trisecant identity for the Jacobian.

Theorem 6.7 (Fay's quadrisecant identity [Fay89])**.** *Let $t \in Pr(\widehat{C}, \sigma)$, $x \in \widehat{C}$ and suppose $\alpha, \beta, \gamma \in \widehat{C}$. Then*

$$\begin{aligned}
&\eta(t - I_P(\beta) - I_P(\gamma))\eta(t + I_P(x) - I_P(\alpha))\frac{E(x,\alpha)}{E(x,\overline{\alpha})}\frac{E(\alpha,\overline{\beta})E(\alpha,\overline{\gamma})}{E(\alpha,\beta)E(\alpha,\gamma)} + \textit{cyclic rotations}\\
&= \eta(t)\eta\left(t + I_P(x) - I_P(\alpha) - I_P(\beta) - I_P(\gamma)\right)\frac{E(x,\alpha)E(x,\beta)E(x,\gamma)}{E(x,\overline{\alpha})E(x,\overline{\beta})E(x,\overline{\gamma})},
\end{aligned}$$

where cyclic rotations refers to cyclic rotations of the triple (α, β, γ).

Remark 6.8. The statement of Theorem 6.7 appears without the explicit constants in [Taĭ97, 6.3.1]. The constants can be found from [Fay89, (37)] by letting $n = 2$, $y_1 = x, x_1 = \alpha, x_2 = \beta, x_3 = \gamma$.

6.4 The cokernel map

The first step in the construction of the inverse spectral transform is to write the cokernel map in terms of Prym theta functions. We start by identifying the cokernel line bundle $\widehat{\mathcal{L}}$.

Proposition 6.9. *The cokernel line bundle $\widehat{\mathcal{L}} \cong \mathcal{O}_{\widehat{C}}(\widehat{S})$ so it has degree equal to g. Moreover,*

$$\widehat{S}_v + d(v) \sim \widehat{S}_u + d(u),$$

for all $u, v \in V(G)$.

Proof. From the definition, we have $\mathcal{G}_{v_0} = \mathcal{O}_{\widehat{C}}$, so that t_{v_0} is a holomorphic section of $\mathcal{H}om(\mathcal{G}_{v_0}, \widehat{\mathcal{L}}) \cong \widehat{\mathcal{L}}$. By Proposition 5.10, we have $\operatorname{div}_{\widehat{C}} t_{v_0} = \widehat{S}$, so we get $\widehat{\mathcal{L}} \cong \mathcal{O}_{\widehat{C}}(\widehat{S})$.

Similarly, t_v is a holomorphic section of

$$\begin{aligned} \mathcal{H}om(\mathcal{G}_v, \widehat{\mathcal{L}}) &= \mathcal{H}om(\mathcal{O}_{\widehat{C}}(d(v)), \widehat{\mathcal{L}}) \\ &\cong \widehat{\mathcal{L}} \otimes O_{\widehat{C}}(d(v))^{\vee} \\ &\cong O_{\widehat{C}}(\widehat{S} - d(v)). \end{aligned} \tag{22}$$

Therefore, by Proposition 5.10, $\operatorname{div}_{\widehat{C}} t_v = \widehat{S}_v \sim \widehat{S} - (d(v))$, which implies that $\widehat{S}_v + d(v) \sim \widehat{S}_u + d(u)$ is equal for all $u, v \in V(G)$. □

Each component of the cokernel map is given by a meromorphic section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ with prescribed order of vanishing at infinity. We will now give an explicit formula for this meromorphic section in terms of theta functions on $\operatorname{Prym}(\widehat{C}, \widehat{\sigma})$.

We define the *discrete Abel-Prym map*

$$\begin{aligned} d_P : V(\widetilde{G}) \cup F(\widetilde{G}) &\to \operatorname{Pr}(\widehat{C}, \widehat{\sigma}) \\ d_P &= \frac{1}{2} I_P \circ d. \end{aligned}$$

Since $I_P = \pi_1 \circ I$, using (20) we have

$$\phi \circ d_P = \frac{1}{2} \phi \circ \pi_1 \circ I \circ d = I \circ d. \tag{23}$$

By Corollary 5.12 and Proposition 6.6, $I(\widehat{S})$ is in a translate of the Prym variety, and using (21), the point $e \in \operatorname{Pr}(\widehat{C}, \widehat{\sigma})$ is given by $e = \frac{1}{2}\pi_1 \left(I(\widehat{S}) - \frac{1}{2} I(q_1) - \frac{1}{2} I(q_2) - \frac{1}{2} I(K_{\widehat{C}}) \right)$. Following [Dol07, (3.20)], define for each vertex or face $u \in V(\widetilde{G}) \cup F(\widetilde{G})$ the meromorphic function

$$\begin{aligned} \psi_u(x) &:= \frac{\eta(I_P(x) + d_P(u) - e)\eta(-e)}{\eta(d_P(u) - e)\eta(I_P(x) - e)} e^{\int_{q_1}^{x} \omega_{\widetilde{d}(u)}} \\ &= \frac{\eta(I_P(x) + d_P(u) - e)\eta(-e)}{\eta(d_P(u) - e)\eta(I_P(x) - e)} \frac{E_{\widetilde{d}(u)}(x)}{E_{\widetilde{d}(u)}(q_1)}. \end{aligned} \tag{24}$$

where the equality of the two expressions is due to (17). By Theorem 6.5, we see that

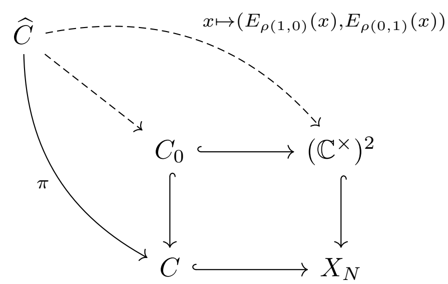
$$\operatorname{div}_{\widehat{C}} \psi_u(x) = \widehat{S}_u - \widehat{S} + \widetilde{d}(u),$$

where $\widehat{S}_u$ is the degree g effective divisor such that $\widetilde{S}_u + \widetilde{d}(u) \sim \widehat{S}$, so ψ_u defines a meromorphic section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ with divisor $\widehat{S}_u + \widetilde{d}(u)$ (cf. A.1.5).

Remark 6.10. The first form in 24 is the form that appears in [Dol07], while the second is, up to normalization, the form that appeared in earlier versions of this article.

We describe the normalization map π explicitly in terms of the prime form. Recall the embedding $\rho : H_1(\mathbb{T}, \mathbb{Z}) \to \mathbb{Z}^Z$.

Lemma 6.11. *The following diagram commutes (dashed arrows indicate rational maps, i.e., maps that are only defined on a dense open subset of $\widehat{C}$):*



Proof. The functions z and w on $(\mathbb{C}^\times)^2$ restrict to functions i^*z and i^*w on C_0, which are meromorphic functions on C, and pull back to meromorphic functions π^*i^*z and π^*i^*w on $\widehat{C}$. By Proposition 5.1, we have

$$\operatorname{div}_{\widehat{C}} \pi^* i^* z = \operatorname{div}_{\widehat{C}} E_{\rho(1,0)}(x),$$

so they are the same up to multiplication by a constant. Since $E_{\rho(1,0)}(q_1) = \pi^*i^*z(q_1) = 1$, the multiplicative constant is 1, and therefore, we have $\pi^*i^*z = E_{\rho(1,0)}(x)$. By the same argument applied to w, we get $\pi^*i^*w = E_{\rho(0,1)}(x)$. ☐

Corollary 6.12. *The functions ψ_u are quasiperiodic with respect to the action of $H_1(\mathbb{T}, \mathbb{Z})$:*

$$\psi_{u+(i,j)} = \pi^* i^* (z^i w^j) \psi_u.$$

Proposition 6.13. *Suppose $e \in Pr(\widehat{C}, \widehat{\sigma})$ is generic and $\widehat{S} := \operatorname{div}_{\widehat{C}} \eta(I_P(x) - e)$ (cf. Proposition 6.5). Then ψ_u is the unique section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ with*

$$\operatorname{div}_{\widehat{C}_0} \psi_u \geq 0,$$
$$\operatorname{div}_{\widehat{C} \cap D_N} \psi_u = \widetilde{d}(u),$$
$$\psi_u(q_1) = 1.$$

Proof. By Corollary 6.3, the section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ satisfying the properties in the statement of the proposition is unique, so it suffices to show that ψ_u has these properties. By Corollary 5.12 and Proposition 6.6, we have

$$I(\widehat{S}) - \frac{1}{2} I(q_1) - \frac{1}{2} I(q_2) - \frac{1}{2} I(K_{\widehat{C}}) \in \phi(\mathrm{Prym}(\widehat{C}, \sigma)),$$

Using (20) and (23), we get $\phi(e) = I(\widehat{S}) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})$. Therefore, by Theorem 6.5 and (23), $\widehat{S}_u := \operatorname{div}_{\widehat{C}} \eta(I_P(x) + d_P(u) - e)$ is a degree g effective divisor satisfying

$$I(\widehat{S}_u) = I(\widehat{S}) - I(\widetilde{d}(u)).$$

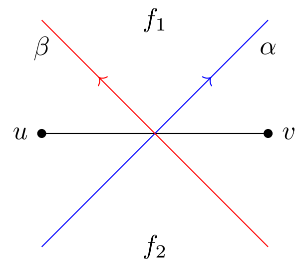


Figure 8: Vertices, faces and zig-zag paths in the definition of the conductance function in (26).

Using (16), we get

$$\operatorname{div}_{\widehat{C}} \psi_u = \widehat{S}_u + \widetilde{d}(u). \tag{25}$$

Plugging in $x = q_1$ in (24) gives $\psi_u(q_1) = 1$.

□

6.5 Construction of the inverse

Now we define the inverse spectral transform. Let uv be an edge in $\widetilde{G}$, f_1 and f_2 be the faces adjacent to uv and let α, β be the zig-zag paths as shown in Figure 8. Define the conductance function [Dol07, (3.24)]

$$\begin{aligned} c(uv) &:= \frac{\eta(d_P(u)-e)\eta(d_P(v)-e)}{\eta(d_P(f_1)-e)\eta(d_P(f_2)-e)} e^{\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}}} \\ &= \frac{\eta(d_P(u)-e)\eta(d_P(v)-e)}{\eta(d_P(f_1)-e)\eta(d_P(f_2)-e)} \frac{E(\alpha,\beta)E(q_1,\overline{\beta})}{E(\alpha,\overline{\beta})E(q_1,\beta)}. \end{aligned} \tag{26}$$

Remark 6.14. We note the similarity of (26) with [GK13, (58)] relating the conductances with the variables that transform like the cube recurrence, which is how we first discovered it before learning of [Dol07].

Lemma 6.15. *The conductance function $c(uv)$ has the following properties:*

1. $c(uv) = c(vu)$;
2. *$c(uv)$ is compatible with taking the dual graph, that is, $c(f_1 f_2) = \frac{1}{c(uv)}$, where $f_1 f_2$ denotes the edge of $G^\vee$ dual to uv.*

Proof. For the first item, notice that exchanging u and v corresponds to letting $\alpha \mapsto \overline{\alpha}, \beta \mapsto \overline{\beta}$ in $e^{\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}}}$. The η-factors are the same, so we need to show that

$$\int_{q_1}^{\overline{\alpha}} \omega_{\overline{\beta}-\beta} = \int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}}.$$

Using the involution σ, we get

$$\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}} = \int_{\overline{q}_1}^{\overline{\alpha}} \sigma^* \omega_{\beta-\overline{\beta}},$$

and now notice that $\overline{q}_1 = q_1$ and that $\sigma^*\omega_{\beta-\overline{\beta}}$ has simple poles with residue 1 (resp., -1) at $\overline{\beta}$ (resp., β). Moreover, the A-periods of $\sigma^*\omega_{\beta-\overline{\beta}}$ are 0 due to $\sigma_* A_i = -A_{i+\frac{g}{2}}$ (18). Since these properties characterize $\omega_{\overline{\beta}-\beta}$, we have $\sigma^*\omega_{\beta-\overline{\beta}} = \omega_{\overline{\beta}-\beta}$.

For the second item, the η-factors are clearly reciprocals, so we need to show that

$$\int_{q_1}^{\overline{\beta}} \omega_{\alpha-\overline{\alpha}} = -\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}}. \tag{27}$$

We have $\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}} = \int_{q_1}^{\overline{\alpha}} \omega_{\overline{\beta}-\beta} = -\int_{q_1}^{\overline{\alpha}} \omega_{\beta-\overline{\beta}} = \int_{\overline{\alpha}}^{q_1} \omega_{\beta-\overline{\beta}}$ (use $\omega_{x-y} = -\omega_{y-x}$ for the second equality). Therefore,

$$\int_{\overline{\alpha}}^{\alpha} \omega_{\beta-\overline{\beta}} = \int_{\overline{\alpha}}^{q_1} \omega_{\beta-\overline{\beta}} + \int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}} = 2\int_{q_1}^{\alpha} \omega_{\beta-\overline{\beta}}.$$

Therefore, (27) is equivalent to

$$\int_{\overline{\beta}}^{\beta} \omega_{\alpha-\overline{\alpha}} = \int_{\overline{\alpha}}^{\alpha} \omega_{\beta-\overline{\beta}},$$

which follows from the "interchange law" $\int_x^y \omega_{b-a} = \int_a^b \omega_{x-y}$ (proved by exponentiating both sides and using (17); see [Fay73, (22)]). □

Proposition 6.16. [Dol07, Corollary 14] *Suppose $e \in Pr(\widehat{C}, \widehat{\sigma})$ is generic and ψ is defined as in (24). Let u, v, f_1, f_2 be the vertices and faces of $\widetilde{G}$ as in Figure 8. Then,*

$$c(uv)(\psi_v - \psi_u) = \psi_{f_2} - \psi_{f_1}. \tag{28}$$

Proof. We normalize so that $\widetilde{d}(u) = 0$. The discrete Abel map in Figure 8 is

$$\widetilde{d}(u) = 0, \quad \widetilde{d}(v) = \overline{\alpha} + \overline{\beta} - \alpha - \beta, \quad \widetilde{d}(f_1) = \overline{\beta} - \beta, \quad \widetilde{d}(f_2) = \overline{\alpha} - \alpha,$$

so that

$$\begin{aligned}
\psi_u(x) &= 1,\\
\psi_v(x) &= \frac{\eta(I_P(x) + d_P(v) - e)\eta(-e)}{\eta(d_P(v) - e)\eta(I_P(x) - e)} \frac{E(x, \overline{\alpha})E(x, \overline{\beta})}{E(x, \alpha)E(x, \beta)} \frac{E(q_1, \alpha)E(q_1, \beta)}{E(q_1, \overline{\alpha})E(q_1, \overline{\beta})},\\
\psi_{f_1}(x) &= \frac{\eta(I_P(x) + d_P(f_1) - e)\eta(-e)}{\eta(d_P(f_1) - e)\eta(I_P(x) - e)} \frac{E(x, \overline{\beta})}{E(x, \beta)} \frac{E(q_1, \beta)}{E(q_1, \overline{\beta})},\\
\psi_{f_2}(x) &= \frac{\eta(I_P(x) + d_P(f_2) - e)\eta(-e)}{\eta(d_P(f_2) - e)\eta(I_P(x) - e)} \frac{E(x, \overline{\alpha})}{E(x, \alpha)} \frac{E(q_1, \alpha)}{E(q_1, \overline{\alpha})}.
\end{aligned}$$

Using $I_P(\alpha) + d_P(f_2) = d_P(u)$ (using $I_P(\overline{\alpha}) = -I_P(\alpha)$, we have $I_P(\alpha) = \frac{1}{2} I_P(\overline{\alpha} - \alpha) = -d_P(f_2)$) etc, we get

$$\lim_{x\to\alpha} \frac{\psi_{f_2}(x)}{\psi_v(x)} = c(uv), \quad \lim_{x\to\beta} \frac{\psi_{f_1}(x)}{\psi_v(x)} = -c(uv). \tag{29}$$

By Proposition 6.13, we know that $c(uv)(\psi_v - \psi_u) - (\psi_{f_2} - \psi_{f_1})$ is a meromorphic section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ and the possible poles are at α and β. However, (29) says that the poles at α (resp., β)

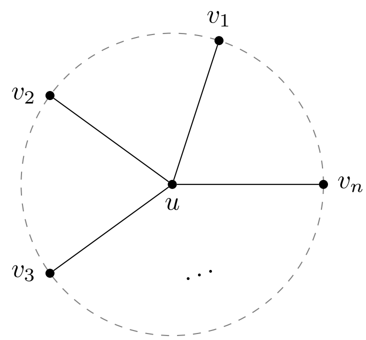


Figure 9: The local configuration near a vertex u

in $c(u,v)\psi_v$ and $-\psi_{f_2}$ (resp., $c(u,v)\psi_v$ and ψ_{f_1}) cancel out, so $c(uv)(\psi_v - \psi_u) - (\psi_{f_2} - \psi_{f_1})$ is a holomorphic section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$. However, by Proposition 6.13, ψ_u is the unique holomorphic section of $\mathcal{O}_{\widehat{C}}(\widehat{S})$ modulo a multiplicative constant, so $c(uv)(\psi_v - \psi_u) - (\psi_{f_2} - \psi_{f_1}) = \text{constant} \cdot \psi_u$. Since $\psi_u(q_1) = 1$ and $(c(uv)(\psi_v - \psi_u) - (\psi_{f_2} - \psi_{f_1}))(q_1) = 0$, the constant is 0. □

Remark 6.17. The equation (28) is simply saying that ψ_u is in the cokernel of the Kasteleyn matrix of the dimer model Γ_G associated to G by Temperley's bijection.

Theorem 6.18. *The rational map $\rho_{G,v_0} : (C, S, \nu) \mapsto (c(uv))_{uv \in E}$ defined in (26) is the inverse of κ_{G,v_0}.*

Proof. 1. $\kappa_{G,v_0} \circ \rho_{G,v_0} = \text{id}$: Let $(C, S, \nu) \in \mathcal{S}'_N$, and let $c = \rho_{G,v_0}(C, S, \nu)$. We wish to show that $\kappa_{G,v_0}(c) = (C, S, \nu)$.

Assume $\kappa_{G,v_0}(c) = (C', S', \nu')$. Let u be a vertex in G and let $v_1, \dots, v_n$ be the vertices adjacent to u in G as shown in Figure 9. Summing the equations (28) for each edge incident to u, we get

$$\sum_{v_k \sim u} c(uv_k)(\psi_u(x) - \phi(v_k u)^{-1} \psi_{v_k}(x)) = 0, \tag{30}$$

where the extra factor of $\phi(v_k u)^{-1}$ is because u, v_k are vertices in G rather than $\widetilde{G}$ (see Corollary 6.12). Therefore, if $x \in C_0$, then $\Delta(z,w)\big|_x$ has nonzero cokernel, so $x \in C'_0$. Since C' and C are two curves with the same Newton polygon that have an infinite number of points in common, $C' = C$.

By (30), the composition $\mathcal{F} \xrightarrow{\widehat{\Delta}} \mathcal{G} \xrightarrow{(\psi_u)_{u \in V}} \mathcal{O}_{\widehat{C}}(\widehat{S})$ is 0. By the universal property of cokernels, there is a unique map of line bundles $\mathcal{O}_{\widehat{C}}(\widehat{S}') \to \mathcal{O}_{\widehat{C}}(\widehat{S})$, which corresponds to a global section of $\mathcal{O}_{\widehat{C}}(\widehat{S} - \widehat{S}')$. Since $\deg \mathcal{O}_{\widehat{C}}(\widehat{S} - \widehat{S}') = 0$ and the only line bundle with degree 0 that has a global section is the trivial line bundle, $\widehat{S} \sim \widehat{S}'$. By Corollary 6.3, $\widehat{S} = \widehat{S}'$, so $S = S'$.

Finally, we check that $\nu = \nu'$. Suppose α is a zig-zag path with $[\alpha] = (i,j)$. Then from (26), we have

$$wt(\alpha) = \prod_{\beta \in Z} E(x, \beta)^{((i,j),[\beta])_{\mathbb{T}}}.$$

By Lemma 6.11,

$$\begin{aligned}(\pi^* i^*(z^i w^j))(x) &= E_{\rho(i,j)}(x) \\ &= \prod_{\beta \in Z} E(x,\beta)^{([\beta],(i,j))_{\mathbb{T}}} \quad \text{(using (16))} \\ &= \prod_{\beta \in Z} E(x,\beta)^{-((i,j),[\beta])_{\mathbb{T}}},\end{aligned}$$

so $(\pi^* i^*(z^i w^j))(\alpha) = \frac{1}{wt(\alpha)}$.

2. $\rho_{G,v_0} \circ \kappa_{G,v_0} = \mathrm{id}$: Suppose c is a conductance function and $\kappa_{G,v_0}(c) = (C, S, \nu)$. Let $v \in V(G)$. By (22), the entry t_v of the matrix of the cokernel map is a holomorphic section of $O_{\widehat{C}}(\widehat{S} - d(v))$, with $\mathrm{div}_{\widehat{C}}\, t_v = \widehat{S}_v$. By Proposition 6.9 and Theorem 6.2, $S' = \widehat{S}_v$. Since $\mathrm{div}_{\widehat{C}}\, \delta_v = d(v)$, we get using (25) that
$$\mathrm{div}_{\widehat{C}}(\delta_v \mapsto \psi_v) = \widehat{S}_v.$$
By Corollary 6.3, t_v is uniquely determined up to multiplication by a nonzero complex number. Since $\Delta(1,1)$ has cokernel map $(1, 1, \ldots, 1)$ (constant functions are discrete harmonic), the normalization is fixed by the requirement that
$$t_v(q_1) = 1.$$
Therefore, the cokernel maps for both c and c' are determined by S and given by $\delta_v \mapsto \psi_v$. Taking transpose, the equation of $\phi^* i^* \Delta^T$ becomes
$$\sum_{v_k \sim u} c(uv_k)(\psi_u(x) - \phi(v_k u)^{-1} \psi_{v_k}(x)) = 0,$$
from which we get that the ratio
$$\frac{c(uv_{k+1})}{c(uv_k)} = -\lim_{x \to \alpha_k} \frac{\phi(v_k u)^{-1} \psi_{v_k}(x)}{\phi(v_{k+1} u)^{-1} \psi_{v_{k+1}}(x)}$$
is determined by $\psi_v, v \in V(G)$. On the other hand, if $c' = \rho_{G,v_0}(C, S, \nu)$, then we also have
$$\sum_{v_k \sim u} c(uv_k)(\psi_u(x) - \phi(v_k u)^{-1} \psi_{v_k}(x)) = 0,$$
so that
$$\frac{c'(uv_{k+1})}{c'(uv_k)} = \frac{c(uv_{k+1})}{c(uv_k)}.$$
It follows that $c' = \text{constant} \cdot c$.

□

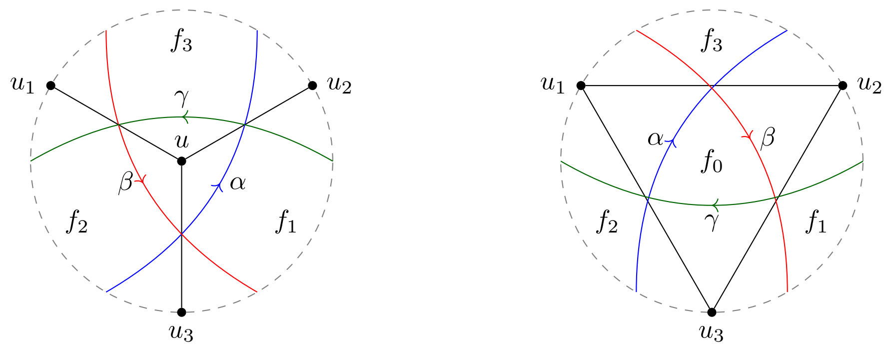


Figure 10: Y-Δ transformation.

7 Y-Δ transformations and Fay's quadrisecant identity

In this section, we will prove Theorem 7.1 which describes how the spectral transform and the Y-Δ transformation interact. The main observation is that the Y-Δ transformation is described by Fay's quadrisecant identity under the spectral transform.

A Y-Δ transformation $s : G_1 \rightsquigarrow G_2$ is induced by sliding a zig-zag path through the crossing of two other zig-zag paths as shown in Figure 10. Therefore, discrete Abel and discrete Abel-Prym maps $d^{G_1}, d_P^{G_1}$ on G_1 induce discrete Abel and discrete Abel-Prym maps $d^{G_2}, d_P^{G_2}$ on G_2. Explicitly, suppose the vertices, faces and zig-zag paths are labeled as shown in Figure 10. By changing the normalization, we assume that $d^{G_1}(u) = 0$. Then, we have

$$\begin{aligned} d^{G_1}(u) &= 0, & d^{G_2}(f_0) &= (\overline{\alpha} - \alpha) + (\overline{\beta} - \beta) + (\overline{\gamma} - \gamma), \\ d^{G_1}(u_1) = d^{G_2}(u_1) &= (\overline{\beta} - \beta) + (\overline{\gamma} - \gamma), & d^{G_1}(f_1) = d^{G_2}(f_1) &= \overline{\alpha} - \alpha, \\ d^{G_1}(u_2) = d^{G_2}(u_2) &= (\overline{\gamma} - \gamma) + (\overline{\alpha} - \alpha), & d^{G_1}(f_2) = d^{G_2}(f_2) &= \overline{\beta} - \beta, \\ d^{G_1}(u_3) = d^{G_2}(u_3) &= (\overline{\alpha} - \alpha) + (\overline{\beta} - \beta), & d^{G_1}(f_3) = d^{G_2}(f_3) &= \overline{\gamma} - \gamma. \end{aligned} \tag{31}$$

The only difference between the vertices and faces of G_1 and G_2 is that the vertex u in G_1 disappears and we get a new face f_0 in G_2. The discrete Abel maps of the vertices u_1, u_2, u_3 and faces f_1, f_2, f_3 which are present on both sides are equal. Therefore, we drop the superscripts in the rest of this section and denote both d^{G_1} and d^{G_2} by d.

Theorem 7.1. *Let $G_1 \rightsquigarrow G_2$ be a Y-Δ transformation with induced map $\mu^s : \mathcal{R}_{G_1} \dashrightarrow \mathcal{R}_{G_2}$ and let v_1 and v_2 be vertices of G_1 and G_2 respectively. The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{R}_N \supset \mathcal{R}_{G_1} & \overset{\mu^s}{\dashrightarrow} & \mathcal{R}_{G_2} \subset \mathcal{R}_N \\ \Big\downarrow{\scriptstyle \kappa_{G,v_1}} & & \Big\downarrow{\scriptstyle \kappa_{G,v_2}} \\ \mathcal{S}'_N & \overset{\xi^s}{\dashrightarrow} & \mathcal{S}'_N \end{array},$$

where the birational map ξ^s is defined as $(C, S_1, \nu_1) \mapsto (C, S_2, \nu_2)$, where

1. *There is a natural bijection between zig-zag paths on G_2 and G_1 induced by the Y-Δ transformation (see Figure 10). ν_2 is obtained by composing this bijection with ν_1.*

2. *S_2 is the (generically unique by Theorem 6.2) degree g effective divisor in C_0 such that*

$$\widehat{S}_2 + d(v_2) \sim \widehat{S}_1 + d(v_1). \tag{32}$$

Moreover, the Y-Δ transformation becomes Fay's quadrisecant identity under the spectral transform.

Proof. The Y-Δ transformation preserves the spectral curve [GK13, Section 5]. The local configuration is shown in Figure 10. Let $e = \frac{1}{2}\pi_1\left(I(\widehat{S}_1) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})\right) + d_P(v_1)$. We show that $\kappa^{-1}_{G_1,v_1} = \kappa^{-1}_{G_2,v_2} \circ \xi^s$. By changing the normalization of d, assume that $d(u) = 0$. From (26), we get

$$\begin{aligned}
a = \kappa^{-1}_{G_1,v_1}(C, S_1, \nu_1)(uu_1) &= \frac{\eta(d_P(u) - e)\eta(d_P(u_1) - e)}{\eta(d_P(f_2) - e)\eta(d_P(f_3) - e)} \frac{E(\gamma, \beta)}{E(\gamma, \overline{\beta})} \frac{E(q_1, \overline{\beta})}{E(q_1, \beta)},\\
b = \kappa^{-1}_{G_1,v_1}(C, S_1, \nu_1)(uu_2) &= \frac{\eta(d_P(u) - e)\eta(d_P(u_2) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_3) - e)} \frac{E(\alpha, \gamma)}{E(\alpha, \overline{\gamma})} \frac{E(q_1, \overline{\gamma})}{E(q_1, \gamma)},\\
c = \kappa^{-1}_{G_1,v_1}(C, S_1, \nu_1)(uu_3) &= \frac{\eta(d_P(u) - e)\eta(d_P(u_3) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_2) - e)} \frac{E(\beta, \alpha)}{E(\beta, \overline{\alpha})} \frac{E(q_1, \overline{\alpha})}{E(q_1, \alpha)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{2}\pi_1\left(I(\widehat{S}_2) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})\right) + d_P(v_2)\\
&= \frac{1}{2}\pi_1\left(I(\widehat{S}_1 + d(v_1) - d(v_2)) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})\right) + d_P(v_2) \quad \text{(using Theorem 6.1 and (32))}\\
&= \frac{1}{2}\pi_1\left(I(\widehat{S}_1) - \frac{1}{2}I(q_1) - \frac{1}{2}I(q_2) - \frac{1}{2}I(K_{\widehat{C}})\right) + d_P(v_1) \quad \left(\text{using } d_P = \frac{1}{2}I_P \circ d = \frac{1}{2}\pi_1 \circ I \circ d\right)\\
&= e.
\end{aligned}$$

Therefore, using (26), we get

$$A = (\kappa^{-1}_{G_2,v_2} \circ \xi^s)(C, S_2, \nu_2)(u_2u_3) = \frac{\eta(d_P(u_2) - e)\eta(d_P(u_3) - e)}{\eta(d_P(f_0) - e)\eta(d_P(f_1) - e)} \frac{E(\gamma, \overline{\beta})}{E(\gamma, \beta)} \frac{E(q_1, \beta)}{E(q_1, \overline{\beta})}.$$

Letting $t = d_P(u) - e$ and $x = q_1$ in the quadrisecant identity (Theorem 6.7), we get

$$\begin{aligned}
&\eta(d_P(u) - I_P(\beta) - I_P(\gamma) - e)\eta(d_P(u) - I_P(\alpha) - e)\frac{E(q_1, \alpha)E(\alpha, \overline{\beta})E(\alpha, \overline{\gamma})}{E(q_1, \overline{\alpha})E(\alpha, \beta)E(\alpha, \gamma)} + \text{cyclic rotations}\\
&= \eta(d_P(u) - e)\eta(d_P(u) - I_P(\alpha) - I_P(\beta) - I_P(\gamma) - e)\frac{E(q_1, \alpha)E(q_1, \beta)E(q_1, \gamma)}{E(q_1, \overline{\alpha})E(q_1, \overline{\beta})E(q_1, \overline{\gamma})}.
\end{aligned}$$

From (31), we have $d_P(u) - I_P(\beta) - I_P(\gamma) = d_P(u_1)$, $d_P(u) - I_P(\alpha) = d_P(f_1)$ etc. Multiplying both sides by $\frac{\eta(d_P(u) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_2) - e)\eta(d_P(f_3) - e)} \frac{E(\alpha, \beta)E(\beta, \gamma)E(\gamma, \alpha)}{E(\alpha, \overline{\beta})E(\beta, \overline{\gamma})E(\gamma, \overline{\alpha})}$, we get

$$a + b + c = \frac{\eta(d_P(u) - e)^2\eta(d_P(f_0) - e)}{\eta(d_P(f_1) - e)\eta(d_P(f_2) - e)\eta(d_P(f_3) - e)} \frac{E(\alpha, \beta)E(\beta, \gamma)E(\gamma, \alpha)}{E(\alpha, \overline{\beta})E(\beta, \overline{\gamma})E(\gamma, \overline{\alpha})} \frac{E(q_1, \alpha)E(q_1, \beta)E(q_1, \gamma)}{E(q_1, \overline{\alpha})E(q_1, \overline{\beta})E(q_1, \overline{\gamma})}.$$

Plugging in these formulas into $\frac{bc}{a+b+c}$, we see that $\frac{bc}{a+b+c} = A$, which is the equation of the Y-Δ transformation (5). ☐

8 Algebro-geometric integrability

The resistor network cluster variety $\mathcal{R}_N$ has a large group of cluster automorphisms arising from the Y-Δ transformation. Each such automorphism defines a discrete dynamical system on $\mathcal{R}_N$. In this section, we prove Theorem 8.1 which states that these discrete dynamical systems are integrable in the algebro-geometric sense.

Consider a sequence T of Y-Δ moves

$$T = \left(G = G_0 \overset{s_1}{\rightsquigarrow} G_1 \overset{s_2}{\rightsquigarrow} \cdots \overset{s_{n-1}}{\rightsquigarrow} G_{n-1} \overset{s_n}{\rightsquigarrow} G_n = G\right),$$

starting and ending with the same graph G. Let $\mu^T := \mu^{s_1} \circ \mu^{s_2} \circ \cdots \circ \mu^{s_n} : \mathcal{R}_G \dashrightarrow \mathcal{R}_G$ be the induced birational map of conductances. Since $\mathcal{R}_G$ is Zariski-dense in $\mathcal{R}_N$, we consider μ^T as a birational automorphism of $\mathcal{R}_N$. It is analogous to a cluster modular transformation as defined in [FG09]. Similarly, let $\xi^T := \xi^{s_1} \circ \xi^{s_2} \circ \cdots \circ \xi^{x_n}$, where ξ^s is as in Theorem 7.1. Let $d = d_0$ denote a discrete Abel map on G. As explained in Section 10, each Y-Δ transformation $s_i : G_{i-1} \rightsquigarrow G_i$ induces a discrete Abel map d_i on G_i from d_{i-1}, so after the sequence T, we obtain a discrete Abel map $d^T = d_n$ on G such that for each $v \in V$, $d^T(v) - d(v)$ is a degree 0 divisor supported on the points at infinity. Composing the diagrams in Theorem 7.1 for each s_i, we get:

Theorem 8.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{R}_N & \overset{\mu^T}{\dashrightarrow} & \mathcal{R}_N \\ \Big\downarrow{\scriptstyle \kappa_{G,v}} & & \Big\downarrow{\scriptstyle \kappa_{G,v}} \\ \mathcal{S}'_N & \overset{\xi^T}{\dashrightarrow} & \mathcal{S}'_N \end{array},$$

where the birational map ξ^T is given by $(C, S, \nu) \mapsto (C, S^T, \nu^T)$ where S^T is the (generically) unique degree g effective divisor satisfying

$$S^T + d^T(v) \sim S + d(v),$$

and ν^T is obtained from ν by composing with the bijection between zig-zag paths induced by T.

For a fixed C, the fiber of the projection $(C, S, \nu) \mapsto C$ over C is a finite cover of the space of degree g effective divisors on C satisfying (9), which is birational to a finite cover of $\mathrm{Prym}(\widehat{C}, \sigma)$. Therefore, along with Proposition 6.6, Theorem 8.1 tells us that the discrete integrable system arising from T is linearized on a finite cover of $\mathrm{Prym}(\widehat{C}, \sigma)$.

9 Further questions

We end by listing some directions that we believe deserve further study.

1. Liouville integrability: Goncharov and Kenyon [GK13] proved that the dimer cluster variety is an algebraic integrable system, with its cluster Poisson structure. We expect the same to be true for the resistor network cluster variety. Find a Poisson structure compatible with the Y-Δ transformation that makes the resistor network cluster variety an algebraic integrable system and with respect to which the fibration by Prym varieties given by the spectral transform is Lagrangian. More generally, the Y-Δ move belongs to the framework of Lam and Pylyavskyy's Laurent phenomenon algebras [LP16], for which we can ask the same question.

2. Massive Laplacian: Boutillier, de Tilière and Raschel [BdTR17] proved analogous results for the massive Laplacian in the isoradial case, that is in the case where the spectral curve has genus one. We expect that there is a common generalization of their results and this paper to the massive Laplacian where the spectral curve has arbitrary genus. We speculate that the massive Y-Δ move might be described by a generalization of the Beauville-Debarre quadrisecant identity [BD87].

3. Relation to the dimer spectral transform: Let G be a minimal resistor network, Γ_G be the associated bipartite graph. Recall the dimer spectral data $\kappa_{\Gamma_G,v} : \mathcal{X}_N \to \mathcal{S}_N$ as defined in [GK13, Proposition 7.2]. By [GK13, Theorem 1.4] or [Foc15], $\kappa_{\Gamma_G,v}$ is birational. We conjecture that the map t that makes the diagram below commute is $(C, S, \nu) \mapsto (C, S + (1,1), \nu)$.

$$\begin{array}{ccc} \mathcal{R}_N & \overset{\kappa_{G,v}}{\dashrightarrow} & \mathcal{S}'_N \\ \big\downarrow & & \big\downarrow t \\ \mathcal{X}_N & \overset{\kappa_{\Gamma_G,v}}{\dashrightarrow} & \mathcal{S}_N \end{array}$$

4. Connections to representation theory: Fock and Marshokov [FM16] showed that the dimer integrable sytems coincide with integrable systems on the Poisson-Lie groups $\widehat{\mathrm{PGL}}$. Is there an analogous construction for resistor networks? We expect that such a construction will relate to the electrical Lie group of Lam and Pylyavskyy [LP15].

5. The Ising model: The dimer cluster variety has another isotropic subvariety corresponding to the Ising model, embedded by Dubédat's bosonization construction [Dub11], which is known to be related to the discrete CKP equation [AGPR20]. Define a spectral transform for the Ising model, and prove its Liouville integrability

A Appendix

A.1 Divisors, line bundles and invertible sheaves

A.1.1 Sheaves of $\mathcal{O}_C$-modules

Let C be a Riemann surface. Let $\mathcal{O}_C$ denote the sheaf of holomorphic/regular functions on C. For $U \subset C$ open,

$$\mathcal{O}_C(U) := \{f : U \to \mathbb{C} \text{ holomorphic}\},$$

with the restriction maps $\mathcal{O}_C(U) \to \mathcal{O}_C(V), f \mapsto f\big|_V$ for $V \subseteq U$ open. A sheaf $\mathcal{F}$ on C is called a *sheaf of $\mathcal{O}_C$-modules* if for every $U \subset C$ open, there is an action of $\mathcal{O}_C(U)$ on $\mathcal{F}(U)$ that is compatible with restriction: for $V \subset U$ open, the diagram

$$\begin{array}{ccc} \mathcal{O}_C(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_C(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commutes. A sheaf of $\mathcal{O}_C$-modules $\mathcal{F}$ is called *invertible* if for every $x \in C$ there exists an open $U \subset C$ such that $\mathcal{F}(U) \cong \mathcal{O}_C(U)$ as sheaves of $\mathcal{O}_C(U)$-modules. An isomorphism $\mathcal{F}(U) \cong \mathcal{O}_C(U)$ is called a *trivialization* of $\mathcal{F}$ over U.

A.1.2 Line bundles

A *line bundle* L on C is a map $\pi : L \to C$ such that

1. For each $x \in C$, the fiber $\pi^{-1}(x)$ is a one-dimensional $\mathbb{C}$-vector space;
2. For every $x \in C$, there is an open neighbourhood U containing x and a homeomorphism $\phi : U \times \mathbb{C} \to \pi^{-1}(U)$ over U that is an isomorphism of $\mathbb{C}$-vector spaces over every $x \in U$, and such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xleftarrow{\quad\phi\quad} & U \times \mathbb{C} \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

   commutes, where the map $\mathrm{pr}_1 : U \times \mathbb{C} \to U$ is projection to the first factor. The map ϕ is called a *trivialization* of L over U.

If ϕ_1 and ϕ_2 are trivializations of L over U_1 and U_2, then they are related over $U_1 \cap U_2$ by an element $g_{12} = \phi_1 \circ \phi_2^{-1}$ of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. If $\{U_i\}$ is an open cover of C, then the functions g_{ij} are called *transition functions* and they satisfy the cocycle condition

$$g_{ij} \circ g_{jk} = g_{ik}$$

over $U_i \cap U_j \cap U_k$. On the other hand, given a cover $\{U_i\}$ and transition functions g_{ij} satisfying the cocycle condition, the line bundle can be recoverd up to isomorphism by gluing together $U_i \times \mathbb{C}$ using g_{ij}.

Given two line bundles L_1, L_2 over C, let $\{U_i\}$ be an open cover over which L_1 and L_2 are both trivialized, and let g_{ij}, h_{ij} be their transition functions. The *tensor product* line bundle $L_1 \otimes L_2$ is obtained as follows: Over U_i, we have $U_i \times (\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) \cong U_i \times \mathbb{C}$ using the canonical isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}, v \otimes w \mapsto v \cdot w$. Under this isomorphism, the transition functions $g_{ij} \otimes h_{ij}$ become $g_{ij} \cdot h_{ij}$.

Line bundles modulo isomorphisms with tensor product form a group called the *Picard group* of C, denoted by $\mathrm{Pic}(C)$.

A.1.3 The invertible sheaf of a line bundle

A *holomorphic/regular section* of L over an open $U \subset C$ is a function $s : U \to L$ such that $\pi \circ s = \mathrm{id}_U$ that is $s(x)$ is in the fiber of L over x for all $x \in U$. Let $\mathcal{O}\{L\}(U) := \{s : U \to L : \pi \circ s = \mathrm{id}_U\}$ denote the set of holomorphic sections of L over U. Since each fiber $\pi^{-1}(x)$ is a $\mathbb{C}$-vector space, we have an action of $\mathcal{O}_C(U)$:

$$\begin{aligned} \mathcal{O}_C(U) \times \mathcal{O}\{L\}(U) &\to \mathcal{O}\{L\}(U) \\ (f, s) &\mapsto f \cdot s, \end{aligned}$$

which makes $\mathcal{O}\{L\}$ a sheaf of $\mathcal{O}_C$-modules. A trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{C}$ of L gives a trivialization $\mathcal{O}\{L\}(U) \cong \mathcal{O}_C(U), s \mapsto f$ where $f : U \to \mathbb{C}$ is the function $\mathrm{pr}_2 \circ \phi \circ s$, where pr_2 is the projection $U \times \mathbb{C} \to \mathbb{C}$ onto the second factor. Therefore, $\mathcal{O}\{L\}$ is an invertible sheaf.

The definition of the tensor product of invertible sheaves is more subtle and involves sheafification, so we refer the reader to [Mir95, Lemma 1.9]. Invertible sheaves modulo isomorphism with tensor product form a group which we denote $\mathrm{Inv}(C)$.

Proposition A.1. *The construction $L \mapsto \mathcal{O}\{L\}$ is an isomorphism of groups $\mathrm{Pic}(C) \to \mathrm{Inv}(C)$.*

Let us briefly describe the inverse map. If $\mathcal{F}$ is an invertible sheaf, let $\{U_i\}$ be an open cover on which it is trivialized: there are isomorpisms $\phi_i : \mathcal{F}(U_i) \to \mathcal{O}_C(U_i)$. Then over $U_i \cap U_j$, we have the isomorphism $\mathcal{O}_C(U_i \cap U_j) \xrightarrow{\phi_i \circ \phi_j^{-1}} \mathcal{O}_C(U_i \cap U_j)$. The functions $g_{ij} = \phi_i \circ \phi_j^{-1}(1)$ are transition functions for the line bundle.

A *holomorphic/regular/global section* of L is an element of $\mathcal{O}\{L\}(C)$. The $\mathbb{C}$-vector space of holomorphic sections of L is denoted by $H^0(C, \mathcal{O}\{L\})$. Let $\{U_i\}$ be an open cover of C with trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$. Let $f_i = \mathrm{pr}_2 \circ \phi_i \circ s\big|_{U_i} \in \mathcal{O}_C(U_i)$. If g_{ij} are the transition functions, then $f_j = f_i g_{ij}$ on $U_i \cap U_j$. On the other hand, if we are given a collection of holomorphic functions f_i over U_i satisfying the $f_j = f_i g_{ij}$, we can glue them to get a global section of L.

A.1.4 Rational sections and divisors

Let L be a line bundle with transition functions g_{ij} with respect to an open cover $\{U_i\}$. A *meromorphic/rational section* t of a L over C is a collection (t_i) of meromorphic functions $t_i : U_i \to \mathbb{C}$ satisfying

$$t_j = t_i g_{ij} \text{ on } U_i \cap U_j \text{ for every } i, j.$$

The *order of vanishing* $\mathrm{ord}_x(t)$ of t at $x \in C$ is the order of vanishing of the rational function t_i at x. The *divisor* of t is

$$\mathrm{div}\ t = \sum_{x \in C} \mathrm{ord}_x(t).$$

Example A.2. If $L = C \times \mathbb{C}$ is the trivial line bundle, then $\mathcal{O}\{L\} = \mathcal{O}_C$ is the sheaf of holomoprhic functions, and a meromorphic section of L is a meromorphic function. The divisor of a meromorphic function is called a *principal divisor.*

Two divisors D and E on C are said to be *linearly equivalent*, and written $D \sim E$, if $D - E$ is a principal divisor. Divisors in C modulo principal divisors, with addition, form a group called the *divisor class group* of C and denoted $\mathrm{Cl}(C)$. If s and t are two meromorphic sections of a line bundle L, then $\mathrm{div}\ s \sim \mathrm{div}\ t$ [Mir95, Proposition 2.23]. Therefore, we have a map $\mathrm{Pic}(C) \to \mathrm{Cl}(C)$.

Proposition A.3. *The map $\mathrm{Pic}(C) \to \mathrm{Cl}(C)$ is an isomorphism of groups.*

A.1.5 The invertible sheaf of a divisor

Propositions A.1 and A.3 tell us that the three groups $\mathrm{Pic}(C), \mathrm{Inv}(C)$ and $\mathrm{Cl}(C)$ are isomorphic. We now explain how the get an invertible sheaf directly from a divisor. Associated to a divisor D on C is a sheaf

$$\mathcal{O}_C(D)(U) := \{t \in K(C)^\times : \mathrm{div}\,|_U t + D|_U \geq 0\} \cup \{0\}, \text{ for all } U \subset C \text{ open},$$

where $K(C)^\times$ denotes the space of meromorphic functions on C. Let p be a point of D with coefficient $a_p \in \mathbb{Z}$ and let U be an open in C containing no other points of D. Let z be rational function on C vanishing at p to order 1 at p and with no other zeros and poles in U. Then

$$\begin{aligned} \mathcal{O}_C(D)(U) &\to \mathcal{O}_C(U) \\ t &\mapsto t \cdot z^{a_p} \end{aligned}$$

is a trivialization of $\mathcal{O}_C(D)$ over U, and therefore, $\mathcal{O}_C(D)$ is an invertible sheaf. Let us describe how to recover the divisor from the invertible sheaf. Let $D = \sum_{i=1}^n a_i p_i$ be a divisor, $\mathcal{O}_C(D)$ the invertible sheaf and L the associated line bundle. Let $\{U_i\}$ be an open cover of C such that each U_i contains exactly point of D (we may need to add some points with $a_i = 0$), and let z_i the local parameters as above, so that we have trivializations $\mathcal{O}_C(D)(U_i) \to \mathcal{O}_C(U_i), t_i \mapsto t_i \cdot z^{a_{p_i}}$. Then a meromorphic function $t \in K(C)^\times$ gives a meromorphic section $\widetilde{t} = (t \cdot z^{a_{p_i}})$ of L. The divisor of this meromorphic section is

$$\mathrm{div}\,\widetilde{t} = \mathrm{div}\,t + D.$$

In particular if t is the constant rational function 1, then $\mathrm{div}\,\widetilde{t} = D$. The section $\widetilde{t}$ is holomorphic if $\mathrm{div}\,t + D \geq 0$.

A.1.6 The determinant line bundle

Suppose $V = \bigoplus_{k=1}^n L_k$ is a vector bundle of rank n, where each L_k is a line bundle with transition functions g_{ij}^k, so that V has transition functions given by the diagonal $n \times n$ matrix

$$h_{ij} = \begin{bmatrix} g_{ij}^1 & & & \\ & g_{ij}^2 & & \\ & & \ddots & \\ & & & g_{ij}^n \end{bmatrix}.$$

The line bundle $\bigwedge^n V$ is called the *determinant line bundle* of V, and it has transition functions $\det h_{ij} = \prod_{i=1}^n g_{ij}^n$, which coincide with the transition functions of $\bigotimes_{i=1}^n L_i$. Therefore, we have:

Proposition A.4. $\bigwedge^n V \cong \bigotimes_{i=1}^n L_i$.

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University of Michigan, Department of Mathematics, 2844 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA

E-mail address: `georgete at umich.edu`