THE TWIST FOR ELECTRICAL NETWORKS AND THE INVERSE PROBLEM

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Abstract. We construct an electrical-network version of the twist map for the positive Grassmannian, and use it to solve the inverse problem of recovering conductances from the response matrix. Each conductance is expressed as a biratio of Pfaffians as in the inverse map of Kenyon and Wilson; however, our Pfaffians are the more canonical B variables instead of their tripod variables, and are coordinates on the positive orthogonal Grassmannian studied by Henriques and Speyer.

1. Introduction

Let $\Gamma = (B \sqcup W, E, F)$ be a planar bipartite graph embedded in a disk $\mathbb{D}$ with vertices $\{d_1, \dots, d_n\}$ on the boundary of $\mathbb{D}$ (and with strand permutation $\pi_{k,n}$; see Section 2.2). Associated with Γ is the space $\mathcal{X}_\Gamma$ of edge weights modulo gauge equivalence. Postnikov [Pos06] constructed a parameterization of the totally positive Grassmannian $\mathrm{Gr}_{>0}(k,n)$ using a map $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathrm{Gr}_{>0}(k,n)$ called *boundary measurement*, where $k := \#W - \#B$. There is another space $\mathcal{A}_\Gamma$ of functions $A : F(\Gamma) \to \mathbb{R}_{>0}$. Scott [Sco06] constructed a function $\Phi_\Gamma : \mathrm{Gr}_{>0}(k,n) \to \mathcal{A}_\Gamma/\mathbb{R}_{>0}$ assigning to each face of Γ a certain Plücker coordinate. The spaces $\mathcal{A}_\Gamma$ and $\mathcal{X}_\Gamma$ are the (positive points of the) $\mathcal{A}$ and $\mathcal{X}$ cluster tori of Fock and Goncharov [FG09], and there is a canonical map $p_\Gamma : \mathcal{A}_\Gamma \to \mathcal{X}_\Gamma$ that assigns to an edge incident to faces f, g the weight $\frac{1}{A_f A_g}$ (with some modification for boundary edges). Muller and Speyer, generalizing earlier work of Berenstein, Fomin and Zelevinsky [BFZ96], and Marsh and Scott [MS16], construct automorphisms $\vec{\tau}$ and $\overleftarrow{\tau}$ of $\mathrm{Gr}_{>0}(k,n)$, called *right* and *left twists*, that sit in the following commutative diagram (where $\sim$ denotes homeomorphism):

$$\begin{array}{ccc} \mathcal{A}_\Gamma/\mathbb{R}_{>0} & \xrightarrow[\sim]{p_\Gamma} & \mathcal{X}_\Gamma \\ \Phi_\Gamma \uparrow \wr & & \wr \downarrow \mathrm{Meas}_\Gamma \\ \mathrm{Gr}_{>0}(k,n) & \underset{\vec{\tau}}{\overset{\overleftarrow{\tau}}{\rightleftarrows}} & \mathrm{Gr}_{>0}(k,n) \end{array}.$$

A key application of the twist is a formula for the inverse boundary measurement map; indeed, $\mathrm{Meas}_\Gamma^{-1} = p_\Gamma \circ \Phi_\Gamma \circ \overleftarrow{\tau}$.

The main goal of this paper is to generalize these results to electrical networks. Let $G = (V, E)$ be a planar graph embedded in a disk $\mathbb{D}$ with vertices $\{b_1, \dots, b_n\}$ on the boundary labeled in clockwise cyclic order. A function $c : E(G) \to \mathbb{R}_{>0}$ is called a *conductance*, and a pair (G, c) is called an *electrical network*. Let $\mathcal{R}_G := \mathbb{R}_{>0}^{E(G)}$ denote the space of conductances on G. In this paper, we focus on *well connected* electrical networks (a genericity condition defined in Section 3.1).

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The *Laplacian* on G is the linear operator $\Delta : \mathbb{R}^{V(G)} \to \mathbb{R}^{V(G)}$ defined by

$$(\Delta f)(v) := \sum_{e=uv} c(e)(f(v) - f(u))$$

where the sum is over all edges uv incident to v. A function $f : V(G) \to \mathbb{R}$ is said to be *harmonic* if $(\Delta f)(v) = 0$ for all internal vertices v of G. Given a function $g : \{b_1, \dots, b_n\} \to \mathbb{R}$ on the boundary vertices, there is a unique extension of g to a harmonic function f_g on $V(G)$, called the *harmonic extension* of g. The linear operator $L : \mathbb{R}^{\{b_1,\dots,b_n\}} \to \mathbb{R}^{\{b_1,\dots,b_n\}}$ defined by $L(g) = (-\Delta f_g)\big|_{\{b_1,\dots,b_n\}}$ is called the *response matrix*. It is a negative semidefinite symmetric matrix whose rows and columns sum to 0. The space of response matrices was characterized by Colin de Verdière [CdV94] and further studied in [CdVGV96, CMM94, CIM98]. The map taking an electrical network to its response matrix is the electrical-network analog of the boundary measurement map. However, this is more than an analogy and the two constructions are directly related as we now explain.

The generalized Temperley's bijection of Kenyon, Propp and Wilson [KPW00] associates to each electrical network (G, c) a weighted bipartite graph $(G_+, [\mathrm{wt}_+])$, giving an embedding $j_G^+ : \mathcal{R}_G \hookrightarrow \mathcal{X}_{G_+}$. The graph G_+ has $2n$ boundary vertices and $\#W - \#B = n + 1$. Lam [Lam18] studied the composition $\mathrm{Meas}_{G_+} \circ j_G^+ : \mathcal{R}_G \to \mathrm{Gr}_{>0}(n+1, 2n)$ and showed that the image of $\mathcal{R}_G$ is a linear slice of $\mathrm{Gr}_{>0}(n+1, 2n)$, which was subsequently identified with a positive Lagrangian Grassmannian $\mathrm{IG}^{\Omega}_{>0}(n+1, 2n)$ of points in $\mathrm{Gr}_{>0}(n+1, 2n)$ that are isotropic for a degenerate skew-symmetric bilinear form Ω in [BGKT21, CGS21] (see also [LP15a]). [CGS21, Theorem 1.8] explicitly identifies the space of response matrices with $\mathrm{IG}^{\Omega}_{>0}(n+1, 2n)$. Therefore, in principle, the inverse problem for electrical networks can be solved using the inverse boundary measurement. However, in practice, the result of inverting the boundary measurement yields a weight on G_+ to which one has to apply a complicated gauge transformation to obtain the conductances. The main goal of the paper is to construct a twist map directly for electrical networks without embedding into weighted bipartite graphs.

Like the space $\mathcal{A}_\Gamma$, there is a second space $\mathcal{B}_G$ associated with an electrical network parameterized by the B variables. The space $\mathcal{B}_G$ consists of functions $B : V(G) \sqcup F(G) \to \mathbb{R}_{>0}$, and there is a canonical map $q_G : \mathcal{B}_G \to \mathcal{R}_G$ defined as follows. Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. Define $q_G : \mathcal{B}_G \to \mathcal{R}_G$ by $c(e) := \frac{B_u B_v}{B_f B_g}$ (cf. Equation (56) in [GK13, Section 5.3.1]). The space $\mathcal{B}_G$ arises from the study of the cube recurrence, a nonlinear recurrence introduced by Propp [Pro01] whose solutions were characterized combinatorially by Carroll and Speyer [CS04] (see also [FZ02, LP15b]). The cube recurrence was further studied by Henriques and Speyer [HS10], who related it to the orthogonal Grassmannian $\mathrm{OG}(n+1, 2n)$ of $(n+1)$-dimensional subspaces that are coisotropic for a certain symmetric bilinear form Q. $\mathrm{OG}(n+1, 2n)$ has an embedding in $\mathbb{CP}^{2^{n-1}-1} \times \mathbb{CP}^{2^{n-1}-1}$ giving bihomogeneous coordinates on $\mathrm{OG}(n+1, 2n)$ called *Cartan coordinates* (which have explicit Pfaffian formulas; see Section 3.5). Henriques and Speyer constructed a homeomorphism $\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \xrightarrow{\sim} \mathcal{B}_G$ assigning to each vertex and face of G a Cartan coordinate, where $\widetilde{\mathrm{OG}}(n+1, 2n)$ is the "affine cone" over $\mathrm{OG}(n+1, 2n)$ and $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ is the subset where all Cartan coordinates are positive. Our first main result is the following.

Theorem 1.1 (cf. Theorem 4.4)**.** *There is a map $\vec{\tau}_{\text{elec}}$, which we call the electrical right twist, such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\quad q_G \quad} & \mathcal{R}_G \\ \Psi_G \uparrow \wr & & \wr \downarrow \text{Meas}_{G_+} \circ j_G^+ \\ \widetilde{\text{OG}}_{>0}(n+1,2n) & \xrightarrow{\vec{\tau}_{\text{elec}}} & \text{IG}^{\Omega}_{>0}(n+1,2n) \end{array} .$$

The two spaces on the left of the commutative diagram in Theorem 1.1 have dimension $\binom{n+1}{2}+1$, whereas the two spaces on the right have dimension $\binom{n}{2}$. Therefore, the electrical right twist as defined is not invertible. Our second main result is:

Theorem 1.2 (cf. Theorem 5.7)**.** *There are actions of $\mathbb{R}^{n+1}_{>0}$ on $\mathcal{B}_G$ and $\widetilde{\text{OG}}_{>0}(n+1,2n)$ compatible with Ψ_G such that upon taking quotients, q_G and $\vec{\tau}_{\text{elec}}$ are invertible. The inverse $\overleftarrow{\tau}_{\text{elec}}$ is called the electrical left twist and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{\quad q_G \quad} & \mathcal{R}_G \\ \Psi_G \uparrow \wr & & \wr \downarrow \text{Meas}_{G_+} \circ j_G^+ \\ \widetilde{\text{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0} & \underset{\vec{\tau}_{\text{elec}}, \ \sim}{\overset{\overleftarrow{\tau}_{\text{elec}}, \ \sim}{\rightleftarrows}} & \text{IG}^{\Omega}_{>0}(n+1,2n) \end{array} .$$

As a consequence, we get that the composition $q_G \circ \Psi_G \circ \overleftarrow{\tau}_{\text{elec}}$ solves the inverse problem for electrical networks. We work out the inverse map explicitly when $n = 3$ in Section 6.

The inverse problem for electrical networks was first solved using a recursive procedure by Curtis, Ingerman and Morrow [CIM98] (see also [CM, Joh12, Rus]). More recently, explicit rational formulas were given by Kenyon and Wilson [KW09, KW17]. In the formulas in [KW09, KW17], the conductances are expressed as biratios of certain variables called *tripod variables* which are only defined for special networks called standard networks. The advantage of our construction is that it works for any well connected electrical network and uses the more canonical B variables instead of the tripod variables. We mention that the inverse problem has also been studied in the cylinder [LP12] and the torus [Geo19]. On the torus, the inverse map of [Geo19] also factors through $q_G : \mathcal{B}_G \to \mathcal{R}_G$ (the B variables are certain Prym theta functions), which further advocates for the naturality of our construction.

We end the introduction with some open problems. If the graph G is not well connected, then $\mathcal{R}_G$ parameterizes a smaller electroid cell in $\text{IG}^{\Omega}(n+1,2n)$ which is the intersection of a positroid cell with $\text{IG}^{\Omega}(n+1,2n)$ [Lam18]. Muller and Speyer defined the twist map for all postroid cells which suggests the following problem.

Problem 1.3. Construct a stratified space whose strata are parameterized by $\mathcal{B}_G$ where G varies over move-equivalence classes of reduced graphs with n vertices on the boundary of the disk. Define an electrical twist map that homeomorphically maps the strata to electroid cells in $\text{IG}^{\Omega}(n+1,2n)$.

There is another notion of positive orthogonal Grassmannian introduced in [HWX14] which was used to parameterize the Ising model by Galashin and Pylyavskyy [GP20]. Similarly, there is a positive Lagrangian Grassmannian associated with the cluster side $\mathcal{A}$ of the Ising model, introduced by Kenyon and Pemantle [KP16, KP14] in relation to the Kashaev recurrence [Kas96]. The two notions of positive orthogonal/Langrangian Grassmannian do not agree. Instead, we expect the relationship to be as in the table below, where the two spaces in each row are related by twist.

	cluster $\mathcal{A}$ side	cluster $\mathcal{X}$ side
dimer models	positive Grassmannian [Sco06]	positive Grassmannian [Pos06]
electrical networks	positive orthogonal Grassmannian [HS10]	positive Lagrangian Grassmannian [BGKT21, CGS21]
Ising models	positive Lagrangian Grassmannian [KP16]	positive orthogonal Grassmannian [GP20]

Problem 1.4. Define a twist map for the Ising model relating the positive orthogonal Grassmannian in [GP20] with the positive Lagrangian Grassmannian in [KP14].

We mention that results relating orthogonal and Lagrangian Grassmannians also appear in [Wan22, Wan23], but the connection to the above table is unclear.

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2. Background on the dimer model and the positive Grassmannian

In this section, we review background on the positive Grassmannian, dimer models, and the twist map.

2.1. Grassmannians and Plücker coordinates.

The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. Let $e_1,\dots,e_n$ denote the standard basis of $\mathbb{C}^n$. For $I=\{i_1<i_2<\cdots<i_k\}\in\binom{[n]}{k}$, let $e_I:=e_{i_1}\wedge\cdots\wedge e_{i_k}$. Then, the e_I form a basis for $\bigwedge^k\mathbb{C}^n$. The *Plücker embedding* is the closed embedding $\mathrm{Pl}:\mathrm{Gr}(k,n)\hookrightarrow\mathbb{P}(\bigwedge^k\mathbb{C}^n)$ sending a subspace X spanned by $x_1,\dots,x_k$ to $[x_1\wedge\cdots\wedge x_k]$. The coefficients $\Delta_I(X)$ of e_I in $x_1\wedge\cdots\wedge x_k$ are called *Plücker coordinates*. Following [Wen21], we call $\widetilde{\mathrm{Gr}}(k,n):=\{(X,x)\mid X\in\mathrm{Gr}(k,n),x\in\bigwedge^k X\}$ the *decorated Grassmannian*. Given $(X,x)\in\widetilde{\mathrm{Gr}}(k,n)$, we denote the coefficient of e_I in v by $\Delta_I(X,x)$. Changing the basis multiplies all the Plücker coordinates by a common scalar, so they are well-defined functions on $\widetilde{\mathrm{Gr}}(k,n)$ but not on $\mathrm{Gr}(k,n)$.

Let $\mathrm{Mat}^\circ(k,n)$ denote the space of $k\times n$ matrices of rank k. $\mathrm{GL_k}$ acts on $\mathrm{Mat}^\circ(k,n)$ by left multiplication and we have identifications

$$\mathrm{GL}_k\setminus\mathrm{Mat}^\circ(k,n)\cong\mathrm{Gr}(k,n)\text{ and }\mathrm{SL}_k\setminus\mathrm{Mat}^\circ(k,n)\cong\widetilde{\mathrm{Gr}}(k,n)\tag{2.1}$$

sending the matrix with rows $x_1,\dots,x_k$ to $\mathrm{span}(x_1,\dots,x_k)$ and $(\mathrm{span}(x_1,\dots,x_k),x_1\wedge\cdots\wedge x_k)$ respectively.

Let $\widetilde{\mathrm{Gr}}_{>0}(k,n)$ denote the *positive decorated Grassmannian*, the subset of $\widetilde{\mathrm{Gr}}(k,n)$ where where all Plücker coordinates are positive real numbers, and let $\mathrm{Gr}_{>0}(k,n)$ denote the *positive Grassmannian*, the subset of $\mathrm{Gr}(k,n)$ where the ratio of any two Plücker coordinates is a positive real number.

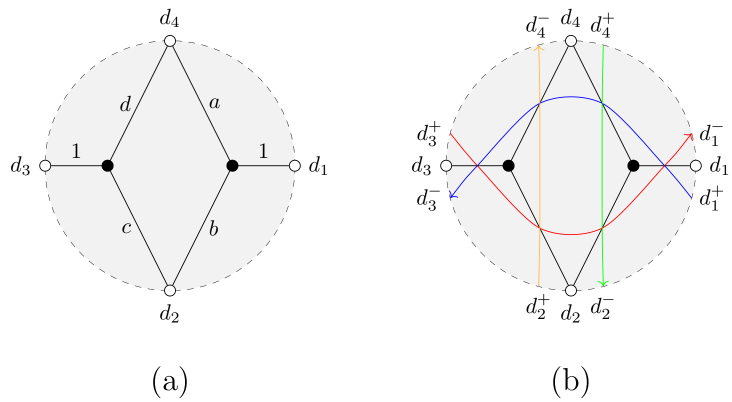


FIGURE 1. A bipartite graph Γ (a) and its medial graph $\Gamma^\times$ with strands (b).

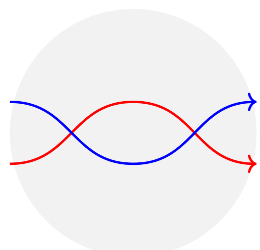
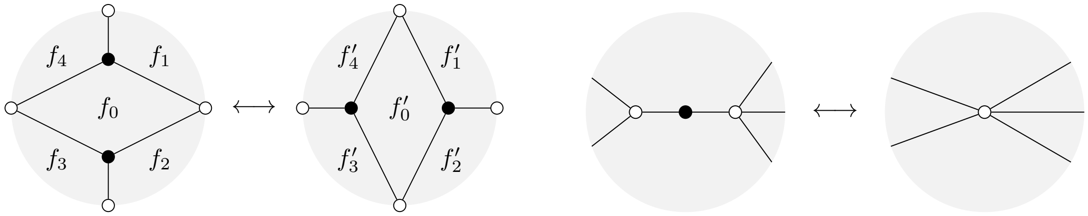


FIGURE 2. A parallel bigon.

2.2. **Planar bipartite graphs in the disk.** Let $\Gamma = (B \sqcup W, E, F)$ be a planar bipartite graph embedded in a disk $\mathbb{D}$ with n vertices on the boundary of $\mathbb{D}$ labeled $d_1, d_2, \ldots, d_n$ in clockwise cyclic order. Here, B denotes the set of black vertices, W the set of white vertices, E the set of edges and F the set of faces respectively. Further, we assume that all the boundary vertices are white. Let $k := \#W - \#B$. The *(oriented) medial graph* $\Gamma^\times$ of Γ is the graph obtained as follows. Place $2n$ vertices of $\Gamma^\times$ labeled $d_1^-, d_1^+, \ldots, d_n^-, d_n^+$ on the boundary of $\mathbb{D}$ such that d_i is between d_i^- and d_i^+. Place a vertex v_e in the middle of each edge e of Γ. Connect v_e and $v_{e'}$ by an edge if they occur consecutively around a face of Γ. For each $i \in [n]$, connect d_i^- (resp., d_i^+) to v_e if e is the last (resp., first) edge in clockwise order incident to d_i. By construction, each d_i^- and each d_i^+ has degree 1 and each v_e degree 4 in $\Gamma^\times$. Orient the edges clockwise around white vertices and counterclockwise around black vertices. Note that this means that edges incident to d_i^- (resp., d_i^+) are oriented towards the outside (resp., inside) of $\mathbb{D}$.

A *strand* of Γ is an oriented walk in $\Gamma^\times$ that either starts and ends at the boundary or is an internal cycle, and at each (degree 4) vertex of the form v_e, the outgoing edge is opposite the incoming one (see Figure 1). We say that Γ is *reduced* (or *minimal*) if:

(1) Each strand starts and ends on the boundary, i.e., no strand path is an internal cycle.
(2) No strand has a self-intersection unless it corresponds to a black leaf incident to a boundary white vertex.
(3) Strands do not form “parallel bigons”, i.e., there is no pair of strands that intersect twice in the same direction (Figure 2).



(a) The spider move. (b) The contraction-uncontraction move.

FIGURE 3. Moves for bipartite graphs.

It is customary to identify a strand with the corresponding oriented walk in Γ that uses the edges e of Γ in the same order that v_e appear in the strand. Note that such a path turns maximally left at white vertices and maximally right at black vertices, and is called a *zig-zag path*.

Let $d^+_{\pi_\Gamma(i)}$ denote the endpoint of the strand that starts at d^-_i. Then, $\pi_\Gamma : [n] \to [n]$ is a permutation called the *strand permutation* of Γ. Let $\pi_{k,n} : [n] \to [n]$ be the permutation $(k+1, k+2, \dots, n, 1, 2, \dots, k-1)$.

Remark 2.1. If $\pi_\Gamma(i) = i$, then we also have to specify a color for i, but this does not occur in $\pi_{k,n}$.

We say that two planar bipartite graphs Γ and Γ' are *move-equivalent* if they are related by the moves shown in Figure 3. Each move $\Gamma \rightsquigarrow \Gamma'$ induces a canonical bijection between $F(\Gamma)$ and $F(\Gamma')$; we denote the face of Γ' corresponding to the face f of Γ by f'. Postnikov [Pos06] and Thurston [Thu17] showed that two reduced bipartite graphs are move-equivalent if and only if they have the same strand permutation.

2.3. **Dimer models and boundary measurement.** Let $\mathrm{wt} : E(\Gamma) \to \mathbb{R}_{>0}$ be a function called an *edge weight*. Two edge weights wt_1 and wt_2 are said to be *gauge equivalent* if there is a function $g : B(\Gamma) \sqcup W(\Gamma) \to \mathbb{R}_{>0}$ that is equal to 1 on the boundary vertices such that for every edge $e = \mathrm{bw}$ with $\mathrm{b} \in B(\Gamma), \mathrm{w} \in W(\Gamma)$, we have $\mathrm{wt}_2(e) = g(\mathrm{b})^{-1}\mathrm{wt}_1(e)g(\mathrm{w})$. Let $\mathcal{X}_\Gamma := \mathbb{R}^{E(\Gamma)}_{>0}/\mathrm{gauge}$ denote the space of edge weights on Γ modulo gauge equivalence. We denote the gauge equivalence class of wt by [wt]. A pair $(\Gamma, [\mathrm{wt}])$ with $[\mathrm{wt}] \in \mathcal{X}_\Gamma$ is called a *dimer model*.

For a face f of Γ with counterclockwise-oriented boundary $\mathrm{w}_1 \xrightarrow{e_1} \mathrm{b}_1 \xrightarrow{e_2} \mathrm{w}_2 \xrightarrow{e_3} \mathrm{b}_2 \xrightarrow{e_4} \cdots \xrightarrow{e_{2k-2}} \mathrm{w}_k \xrightarrow{e_{2k-1}} \mathrm{b}_k \xrightarrow{e_{2k}} \mathrm{w}_1$, let

$$X_f := \prod_{i=1}^{k} \frac{\mathrm{wt}(e_{2i})}{\mathrm{wt}(e_{2i-1})}$$

denote the alternating product of the edge weights around the boundary of f. The X_f's are invariant under gauge equivalence and provide coordinates on $\mathcal{X}_\Gamma$ satisfying the relation $\prod_{f \in F(\Gamma)} X_f = 1$, so $\mathcal{X}_\Gamma \cong \mathbb{R}^{\#F(\Gamma)-1}_{>0}$.

A move $\Gamma \rightsquigarrow \Gamma'$ induces a homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ defined as follows:

(1) Spider move at a face f_0: The homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ is given by

$$X_{f'_0} := \frac{1}{X_{f_0}}, X_{f'_1} := X_{f_1}(1+X_{f_0}), X_{f'_2} := \frac{X_{f_2}}{(1+\frac{1}{X_{f_0}})}, X_{f'_3} := X_{f'_3}(1+X_{f_0}), X_{f_4} := \frac{X_{f'_4}}{(1+\frac{1}{X_{f_0}})},$$

and $X_{f'} := X_f$ for $f' \in F(\Gamma') \setminus \{f'_0, f'_1, f'_2, f'_3, f'_4\}$.

(2) Contraction-uncontraction move: The homeomorphism $\mathcal{X}_\Gamma \xrightarrow{\sim} \mathcal{X}_{\Gamma'}$ is $X_{f'} := X_f$ for all $f' \in F(\Gamma')$.

Given a strand permutation π, let $\mathcal{X}_\pi := \bigsqcup_{\pi_\Gamma = \pi} \mathcal{X}_\Gamma \big/ \text{moves}$ denote the *space of dimer models*, where the union is over all reduced bipartite graphs Γ with strand permutation π.

A *dimer cover* (or *almost perfect matching*) of Γ is a subset of $E(\Gamma)$ that uses each internal vertex of Γ and a subset of the boundary vertices exactly once. The *weight* $\mathrm{wt}(M)$ of a dimer cover M is defined to be $\prod_{e \in M} \mathrm{wt}(e)$. For a dimer cover M, let

$$\partial M := \{i \in [n] \mid d_i \text{ is not used by } M\} \in \binom{[n]}{k},$$

where n and k are as in Section 2.2. For $I \in \binom{[n]}{k}$, define the *dimer partition function*

$$Z_I := \sum_{M \mid \partial M = I} \mathrm{wt}(M).$$

Postnikov [Pos06] defined the *boundary measurement map*

$$\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathbb{P}\big(\textstyle\bigwedge^k \mathbb{C}^n\big)$$

sending $[\mathrm{wt}]$ to $[\sum_{I \in \binom{[n]}{k}} Z_I e_I]$. Meas_Γ is well-defined, since the gauge equivalence multiplies all Z_I's by a scalar. The following theorem is due to Postnikov [Pos06] in a different language (see also [PSW09] and [Lam16, Corollary 7.14]).

Theorem 2.2. *For a reduced* Γ *with* $\pi_\Gamma = \pi_{k,n}$, $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \xrightarrow{\sim} \mathrm{Gr}_{>0}(k,n)$ *is a homeomorphism. If* Γ *and* Γ' *are related by a move, then the following diagram commutes:*

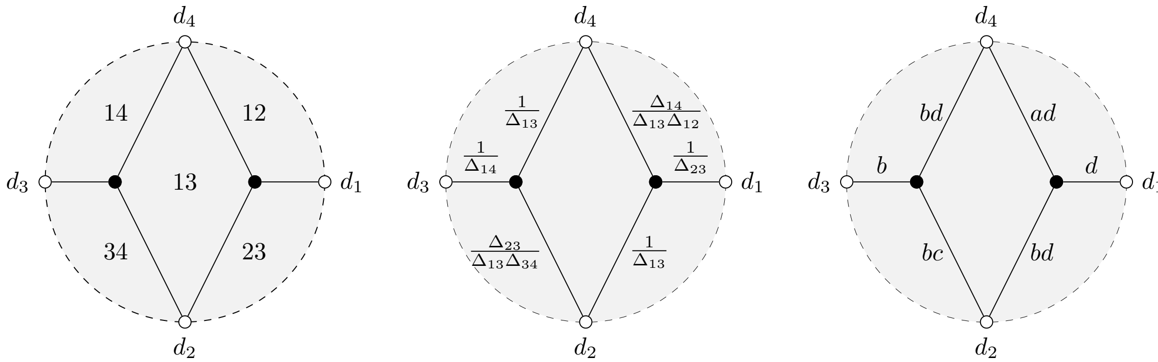
$$\begin{array}{ccc} \mathcal{X}_\Gamma & & \\ {\scriptstyle move}\ \big\downarrow \wr & \searrow^{\mathrm{Meas}_\Gamma}_{\sim} & \\ & & \mathrm{Gr}_{>0}(k,n) \\ & \nearrow^{\sim}_{\mathrm{Meas}_{\Gamma'}} & \\ \mathcal{X}_{\Gamma'} & & \end{array} .$$

Therefore, the maps Meas_Γ *glue to a homeomorphism* $\mathrm{Meas} : \mathcal{X}_{\pi_{k,n}} \xrightarrow{\sim} \mathrm{Gr}_{>0}(k,n)$.

Example 2.3. Let (Γ, wt) be the weighted bipartite graph shown in Figure 1(a). From the strands shown in Figure 1(b), obtain the strand matching to be $\pi_{2,4}$. The boundary measurement map sends $[\mathrm{wt}]$ to $[ae_{12} + (ac+bd)e_{13} + be_{14} + de_{23} + e_{24} + ce_{34}]$, which is the image under Pl of

$$X := \text{row span} \begin{bmatrix} b & 1 & c & 0 \\ -a & 0 & d & 1 \end{bmatrix}. \tag{2.2}$$

2.4. *A* **variables.** Let $\mathcal{A}_\Gamma := \mathbb{R}_{>0}^{F(\Gamma)}$ denote the space of functions $A : F(\Gamma) \to \mathbb{R}_{>0}$. A move $\Gamma \rightsquigarrow \Gamma'$ induces a homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ as follows:



(a) Face labels. (b) $\Phi_\Gamma \circ p_\Gamma$. (c) $p_\Gamma \circ \Phi_\Gamma \circ \overleftarrow{\tau}$.

FIGURE 4. Inverting the boundary measurement map for the graph in Figure 1.

(1) Spider move at a face f_0: The homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ is given by the cluster mutation formula

$$A_{f_0'} := \frac{A_{f_1}A_{f_3} + A_{f_2}A_{f_4}}{A_{f_0}}$$

and $A_{f'} := A_f$ for $f' \in F(\Gamma') \setminus \{f_0'\}$.

(2) Contraction-uncontraction move: The homeomorphism $\mathcal{A}_\Gamma \xrightarrow{\sim} \mathcal{A}_{\Gamma'}$ is $A_{f'} := A_f$ for all $f' \in F(\Gamma')$.

Let $\mathcal{A}_\pi := \bigsqcup_{\pi_\Gamma=\pi} \mathcal{A}_\Gamma \Big/ \text{moves}$.

Remark 2.4. The spaces $\mathcal{X}_\Gamma$ and $\mathcal{A}_\Gamma$ are the positive points of the $\mathcal{X}$ and $\mathcal{A}$ cluster tori associated with Γ respectively (see [FG09]), and $\mathcal{X}_\pi$ and $\mathcal{A}_\pi$ are the positive points of the $\mathcal{X}$ and $\mathcal{A}$ cluster varieties respectively. Since the cluster varieties do not appear directly in this paper, we have chosen to denote the positive points by $\mathcal{X}_\Gamma$ instead of $\mathcal{X}_\Gamma(\mathbb{R}_{>0})$ etc.

Definition 2.5. The faces of the medial graph $\Gamma^\times$ are in bijection with $B(\Gamma) \sqcup W(\Gamma) \sqcup F(\Gamma)$. We say that $f \in F(\Gamma)$ is to the left of a strand if the corresponding face of $\Gamma^\times$ is to the left of the strand. For each face f of Γ, define the *(target) face label*

$$S(f) := \{i \in [n] \mid f \text{ is on the left of the strand ending at } d_i^+\}.$$

For each face f, $S(f)$ is a k-element subset of $[n]$. Let $f_1^-, \dots, f_n^-$ denote the boundary faces of Γ so that f_i^- is between d_{i-1} and d_i. If $\pi_\Gamma = \pi_{k,n}$, then $S(f_i^-) = \{i, i+1, \dots, i+k-1\}$ are the cyclically consecutive subsets.

Example 2.6. For the graph in Figure 1(a), using the strands shown in Figure 1(b), we compute the face labels as shown in Figure 4.

Scott [Sco06] defined the map

$$\Phi_\Gamma : \widetilde{\mathrm{Gr}}_{>0}(k,n) \to \mathcal{A}_\Gamma$$

sending (X,x) to $(\Delta_{S(f)}(X,x))_{f \in F(\Gamma)}$.

Theorem 2.7 (Scott, [Sco06, Theorem 4]). *For every reduced* Γ *with* $\pi_\Gamma = \pi_{k,n}$, $\Phi_\Gamma : \widetilde{\mathrm{Gr}}_{>0}(k,n) \xrightarrow{\sim} \mathcal{A}_\Gamma$ *is a homeomorphism. If* Γ_1 *and* Γ_2 *are related by a move, then*

$$\begin{array}{ccccc} & & \mathcal{A}_{\Gamma_1} & & \\ & \overset{\Phi_{\Gamma_1}}{\nearrow} {\scriptstyle\sim} & & \searrow & \\ \widetilde{\mathrm{Gr}}_{>0}(k,n) & & \sim \Big\downarrow \mathit{move} & & \mathcal{A}_{\pi_{k,n}} \\ & \underset{\Phi_{\Gamma_2}}{\searrow} {\scriptstyle\sim} & & \nearrow & \\ & & \mathcal{A}_{\Gamma_2} & & \end{array}$$

commutes, so we obtain a well-defined homeomorphism $\Phi : \widetilde{\mathrm{Gr}}_{>0}(k,n) \xrightarrow{\sim} \mathcal{A}_{\pi_{k,n}}$.

2.5. **Twist.** We introduce the twist map defined by Marsh and Scott [MS16] and generalized by Muller and Speyer [MS17]. We follow the normalization conventions of [MS17]. Let M be a $k \times n$ matrix whose $k \times k$ minors are all nonzero. For any $i \in [n]$, let M_i denote the ith column of M. We extend this definition to all $i \in \mathbb{Z}$ by defining $M_i := M_{\bar{i}}$ where $\bar{i} \in [n]$ is the reduction of $i \in \mathbb{Z}$ modulo n. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^k$.

Definition 2.8. The *right twist* of M is the $k \times n$ matrix $\vec{\tau}(M)$ whose column $\vec{\tau}(M)_i$ is defined by

$$\langle \vec{\tau}(M)_i, M_i \rangle = 1 \text{ and } \langle \vec{\tau}(M)_i, M_j \rangle = 0 \text{ for } i < j \leq i + k - 1.$$

Similarly, the *left twist* of X is the $k \times n$ matrix $\overleftarrow{\tau}(M)$ whose column $\overleftarrow{\tau}(M)_i$ is defined by

$$\langle \overleftarrow{\tau}(M)_i, M_i \rangle = 1 \text{ and } \langle \overleftarrow{\tau}(M)_i, M_j \rangle = 0 \text{ for } i - k + 1 \leq j < i.$$

Theorem 2.9 (Muller and Speyer, [MS17, Corollary 6.8]). *Under the identifications (2.1), the right and left twists descend to mutually inverse homeomorphisms of* $\widetilde{\mathrm{Gr}}_{>0}(k,n)$ *and* $\mathrm{Gr}_{>0}(k,n)$.

Definition 2.10. We denote the right twist of $(X, x) \in \widetilde{\mathrm{Gr}}_{>0}(k,n)$ (resp., $X \in \mathrm{Gr}_{>0}(k,n)$) by $\vec{\tau}(X, x)$ (resp., $\vec{\tau}(X)$), and similarly for the left twist.

Example 2.11. The left twist of X in (2.2) is $\overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} \frac{1}{b} & 1 & 0 & -\frac{d}{c} \\ 0 & \frac{b}{a} & \frac{1}{d} & 1 \end{bmatrix}$.

Definition 2.12. Let Γ be a reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$. Define the map $p_\Gamma : \mathcal{A}_\Gamma \to \mathcal{X}_\Gamma$ sending A to $[\mathrm{wt}]$ as follows. Let $e \in E(\Gamma)$ be an edge and let $f, g \in F(\Gamma)$ be the two faces incident to e. Define

$$\mathrm{wt}(e) := \begin{cases} \frac{1}{A_f A_g} & \text{if } e \text{ is not incident to a boundary white vertex,} \\ \frac{A_{f_i^-}}{A_f A_g} & \text{if } e \text{ is incident to boundary white vertex } d_i, \end{cases}$$

where f_i^- is the boundary face of Γ between d_{i-1} and d_i.

Theorem 2.13. [MS17, Theorem 7.1 and Remark 7.2] *Let Γ be reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$. The following diagrams commute.*

$$
\begin{array}{ccc}
\mathcal{A}_\Gamma & \xrightarrow{p_\Gamma} & \mathcal{X}_\Gamma \\
\Phi_\Gamma \uparrow \sim & & \sim \downarrow \mathrm{Meas}_\Gamma \\
\widetilde{\mathrm{Gr}}_{>0}(k,n) & \xrightarrow[\vec{\tau}]{} & \mathrm{Gr}_{>0}(k,n)
\end{array}
,\qquad
\begin{array}{ccc}
\mathcal{A}_\Gamma/\mathbb{R}_{>0} & \xrightarrow[\sim]{p_\Gamma} & \mathcal{X}_\Gamma \\
\Phi_\Gamma \uparrow \sim & & \sim \downarrow \mathrm{Meas}_\Gamma \\
\mathrm{Gr}_{>0}(k,n) & \overset{\overleftarrow{\tau}}{\underset{\vec{\tau}}{\rightleftarrows}}\ (\sim) & \mathrm{Gr}_{>0}(k,n)
\end{array}
. \tag{2.3}
$$

In the diagram on the right, the quotient is by the action of $\mathbb{R}_{>0}$ on $\mathcal{A}_\Gamma$ multiplying all the A variables by a scalar.

Remark 2.14. The map p_Γ is an incarnation of the canonical map between $\mathcal{A}$ and $\mathcal{X}$ cluster varieties in Fock and Goncharov [FG09].

Example 2.15. Recall Examples 2.3 and 2.11. The Plücker coordinates of $\overleftarrow{\tau}(X)$ are

$$\Delta_{12} = \frac{1}{a}, \Delta_{13} = \frac{1}{bd}, \Delta_{14} = \frac{1}{b}, \Delta_{23} = \frac{1}{d}, \Delta_{24} = 1 + \frac{bd}{ac}, \Delta_{34} = \frac{1}{c}.$$

The compositions $p_\Gamma \circ \Phi_\Gamma$ and $p_\Gamma \circ \Phi_\Gamma \circ \overleftarrow{\tau}$ are shown in Figure 4(b) and Figure 4(c) respectively. The weights in Figure 1(a) and Figure 4(c) are easily seen to be gauge equivalent.

Definition 2.16. For $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{>0}$ and $X \in \mathrm{Gr}_{>0}(n+1, 2n)$, let $t \cdot X \in \mathrm{Gr}_{>0}(n+1, 2n)$ denote the point obtained as follows. Let M be a $k \times n$ matrix such that X is the row span of M. Then, $t \cdot X$ is the row span of the matrix $t \cdot M$ defined by $(t \cdot M)_i := t_i M_i$.

Let Γ be a reduced bipartite graph with $\pi_\Gamma = \pi_{k,n}$ and let $[\mathrm{wt}] \in \mathcal{X}_\Gamma$. Let $\mathbb{R}^n_{>0}$ act on $\mathcal{X}_\Gamma$ by multiplying the weights of all edges incident to d_i by $\frac{1}{t_i}$. The following lemma is used in the proof of Theorem 4.4.

Lemma 2.17. *The map $\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathrm{Gr}_{>0}(k,n)$ is $\mathbb{R}^n_{>0}$ equivariant.*

Proof. We have $\Delta_I(t \cdot X) = (\prod_{i \in I} t_i)\Delta_I(X)$. On the other hand,

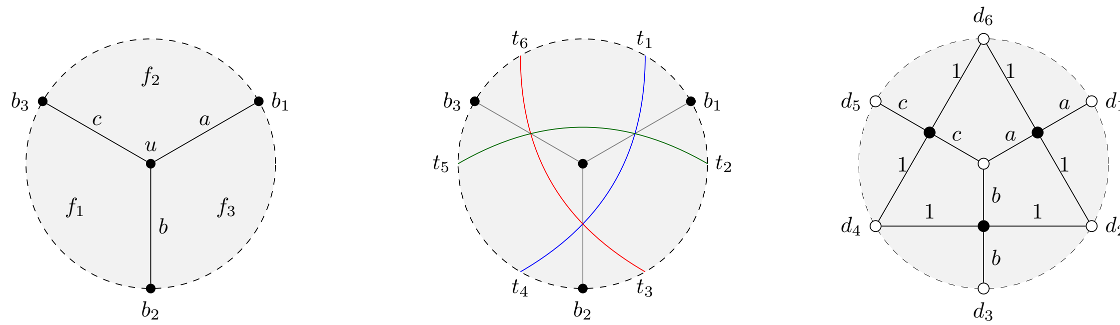
$$
\begin{aligned}
\mathrm{Meas}_\Gamma(t \cdot [\mathrm{wt}]) &= \left[\sum_{I \in \binom{[n]}{k}} \left(\prod_{i \notin I | d_i \in W(\Gamma)} t_i \right) Z_I e_I \right] \\
&= \left[\sum_{I \in \binom{[n]}{k}} \left(\prod_{i \in I} t_i \right) Z_I e_I \right],
\end{aligned}
$$

where in the second equality we rescaled by $\prod_{i \in [n] | d_i \in W(\Gamma)} t_i$. □

The following two properties of the twist will be required later.

Proposition 2.18 (Muller and Speyer, [MS17, (9) in the proof of Proposition 6.6 and Proposition 6.1]). *Let $X \in \mathrm{Gr}_{>0}(k,n)$.*

1. *For any boundary face f_i^-, we have $\Delta_{S(f_i^-)}(\vec{\tau}(X)) = \frac{1}{\Delta_{S(f_i^-)}(X)}$.*
2. *If $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{>0}$, then $\vec{\tau}(t \cdot X) = t^{-1} \cdot \vec{\tau}(X)$, where $t^{-1} := (\frac{1}{t_1}, \dots, \frac{1}{t_n})$.*



(a) An electrical network (G, c). (b) $G^\times$. (c) $(G_+, [\mathrm{wt}_+])$.

FIGURE 5. An electrical network with $n = 3$ and its associated graphs. The three strands of G are given different colors.

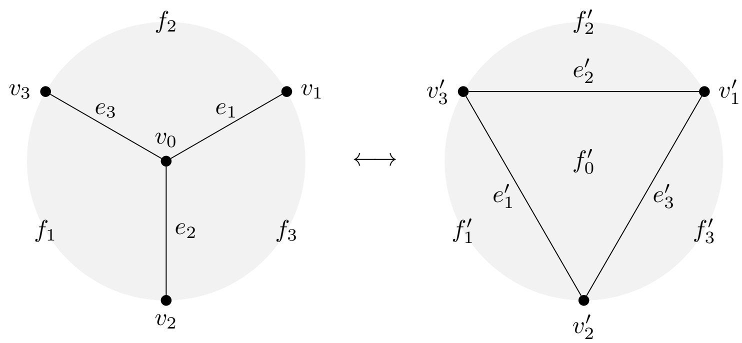


FIGURE 6. The Y-Δ move.

3. Electrical networks

3.1. **Reduced graphs in the disk.** Let $G = (V, E, F)$ be a planar graph embedded in the disk $\mathbb{D}$ with n vertices on the boundary labeled $b_1, b_2, \ldots, b_n$. The *medial graph* $G^\times$ of G is the graph obtained as follows. Place $2n$ vertices of $G^\times$ labeled $t_1, t_2, \ldots, t_{2n}$ on the boundary of $\mathbb{D}$ such that b_i is between t_{2i-1} and t_{2i} and a vertex v_e in the middle of each edge e of G. Connect v_e and $v_{e'}$ by an edge if they occur consecutively around a face of G. For each $i \in [n]$, connect t_{2i-1} (resp., t_{2i}) to v_e if e is the last (resp., first) edge in clockwise order incident to b_i. By construction, each t_i has degree 1 and each v_e degree 4 in $G^\times$. A *strand* of G is a maximal walk in $G^\times$ that goes "straight through" every vertex v_e in it, i.e., if $e_1^\times, e_2^\times$ are two consecutive edges of $G^\times$ in the walk with common vertex v_e, then $e_1^\times$ and $e_2^\times$ are opposite each other with respect to the cyclic order of edges around v_e (which makes sense since v_e has degree 4). Unlike strands in a bipartite graphs, strands in G are unoriented.

Example 3.1. Figure 5(b) shows the medial graph of the electrical network in Figure 5(a).

The graph G is called *reduced* if:

(1) Every strand starts and ends at a boundary vertex, i.e., no strand is an internal cycle.
(2) Strands have no self-intersections.
(3) There is no pair of strands that intersect twice.

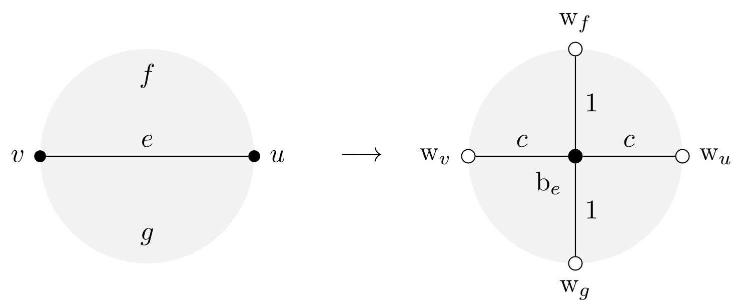


FIGURE 7. The procedure to obtain G_+ from G, where c is the conductance of the edge e.

The *medial pairing* of G is the matching on $[2n]$ defined by

$$\tau_G := \{\{i,j\} \mid \text{there is a strand between } t_i \text{ and } t_j\}.$$

In this paper, we only consider reduced graphs G with medial pairing $\tau_n := \{\{1,n+1\},\{2,n+2\},\ldots,\{n,2n\}\}$; such graphs are called *well connected*. If G is well connected, since the edges of G are in bijection with crossings of strands and any two strands cross exactly once, G has $\binom{n}{2}$ edges.

Example 3.2. For the electrical network in Figure 5(a), the medial graph is shown in Figure 5(b), from which we see that G is reduced with medial pairing τ_3.

We say that G and G' are *move-equivalent* if they are related by a sequence of Y-Δ moves (Figure 6). A Y-Δ move $G \rightsquigarrow G'$ induces canonical bijections $V(G) \sqcup F(G) \xrightarrow{\sim} V(G') \sqcup F(G')$ and $E(G) \xrightarrow{\sim} E(G')$. Two graphs G and G' are move-equivalent if and only if they have the same medial pairing [CdV94].

3.2. **The space of electrical networks and the positive Lagrangian Grassmannian.** Let $c : E(G) \to \mathbb{R}_{>0}$ be a function called *conductance*, and let $\mathcal{R}_G := \mathbb{R}_{>0}^{E(G)} \cong \mathbb{R}_{>0}^{\binom{n}{2}}$ be the space of conductances on G. A pair (G,c) with $c \in \mathcal{R}_G$ is called an *electrical network*.

A Y-Δ move $G \rightsquigarrow G'$ induces a homeomorphism $\mathcal{R}_G \xrightarrow{\sim} \mathcal{R}_{G'}$ given by

$$c(e_1') := \frac{c(e_2)c(e_3)}{C}, c(e_2') := \frac{c(e_1)c(e_3)}{C}, c(e_3') := \frac{c(e_1)c(e_2)}{C},$$

where $C := c(e_1)c(e_2) + c(e_1)c(e_3) + c(e_2)c(e_3)$ and the edges are labeled as in Figure 6, while the conductances of edges not involved in the Y-Δ move are unchanged. Let $\mathcal{R}_n := \bigsqcup_{\tau_G=\tau_n} \mathcal{R}_G \Big/ \text{moves}$ denote the *space of electrical networks*.

The generalized Temperley's bijection of [KPW00] associates a dimer model $(G_+, [\mathrm{wt}_+])$ to (G,c) as follows. Place a black vertex b_e in the middle of every edge e of G, a white vertex w_v at every vertex v of G, a white vertex w_f in the middle of every internal face f, and a white vertex w_f in the middle of the intersection of the boundary of $\mathbb{D}$ with f for every boundary face f of G. If v is a vertex of G incident to edge e, draw an edge $\mathrm{b}_e\mathrm{w}_v$ and assign $\mathrm{wt}_+(\mathrm{b}_e\mathrm{w}_v) := c(e)$. If f is a face of G incident to e, draw an edge $\mathrm{b}_e\mathrm{w}_f$ and assign $\mathrm{wt}_+(\mathrm{b}_e\mathrm{w}_f) := 1$ (see Figure 7). G_+ has $2n$ boundary white vertices which we label $d_1, \ldots, d_{2n}$ in clockwise cylic order as follows:

$$d_{2i-1} := \mathrm{w}_{b_i} \text{ and } d_{2i} := \mathrm{w}_{f_i},$$

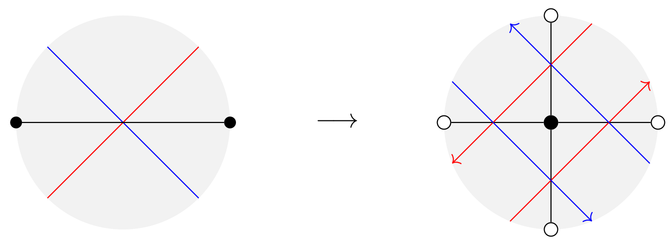


FIGURE 8. The $1:2$ correspondence between strands in G and strands in G_+.

where f_i denotes the boundary face between b_i and b_{i+1}.

An Euler characteristic computation shows that $\#W(G_+) - \#B(G_+) = n+1$. Moreover, there is a $1:2$ correspondence between strands in G and strands in G_+ (Figure 8) which can be seen locally. For $i \in [n]$, let β_i denote the strand in G between t_i and t_{n+i}. Let α_i denote the strand of G_+ ending at d_i^+. From the local picture in Figure 8, we see that α_i starts at d_{n+i-1}^-, and that the two strands in G_+ that correspond to β_i are α_i and α_{n+i}.

Example 3.3. The weighted bipartite graph associated to the electrical network in Figure 5(a) is shown in Figure 5(c).

Remark 3.4. The notation G_+ is inspired by the notation $G_\square$ for the Ising graph in [GP20], since we are replacing each edge of G with a $+$.

The map $(G, c) \mapsto (G_+, [\mathrm{wt}_+])$ defines an inclusion $j_G^+ : \mathcal{R}_G \hookrightarrow \mathcal{X}_{G_+}$.

Proposition 3.5 (Goncharov and Kenyon, [GK13, Lemma 5.11])**.** *If G and G' are related by a Y-Δ move, then there is a sequence of moves for bipartite graphs relating G_+ and G'_+ making the following diagram commute.*

$$\begin{array}{ccc} \mathcal{R}_G & \xhookrightarrow{j_G^+} & \mathcal{X}_{G_+} \\ {\scriptstyle Y\text{-}\Delta\ move}\downarrow\wr & & \wr\downarrow{\scriptstyle moves} \\ \mathcal{R}_{G'} & \xhookrightarrow{j_{G'}^+} & \mathcal{X}_{G'_+} \end{array}.$$

Therefore, the inclusions j_G^+ glue to an inclusion $j^+ : \mathcal{R}_n \to \mathcal{X}_{\pi_{n+1,2n}}$.

Let $\Omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be the degenerate skew symmetric bilinear form

$$\Omega(x,y) = \sum_{i=1}^{n}(x_{2i-1}y_{2i} - x_{2i}y_{2i-1}) + \sum_{i=1}^{n-1}(x_{2i+1}y_{2i} - x_{2i}y_{2i+1}) + (-1)^n(x_1y_{2n} - x_{2n}y_1). \tag{3.1}$$

We say that $X \in \mathrm{Gr}(n+1, 2n)$ is *isotropic* for Ω if $\Omega(x, y) = 0$ for any $x, y \in X$. Let $\mathrm{IG}^\Omega(n+1, 2n)$ be the *Lagrangian Grassmannian* of isotropic subspaces inside $\mathrm{Gr}(n+1, 2n)$ and $\mathrm{IG}^\Omega_{>0}(n+1, 2n) := \mathrm{IG}^\Omega(n+1, 2n) \cap \mathrm{Gr}_{>0}(n+1, 2n)$ the *positive Lagrangian Grassmannian.*

Remark 3.6. The form Ω has a two-dimensional kernel which must be contained in every isotropic $(n+1)$-dimensional subspace. Therefore, if we quotient by the kernel, we get that

$\mathrm{IG}^{\Omega}(n+1,2n)$ is isomorphic to the Lagrangian Grassmannian $\mathrm{LG}(n-1,2n-2)$. The total positivity structure of $\mathrm{IG}^{\Omega}_{>0}(n+1,2n)$ is also non-standard (see [CGS21, Section 5] for further discussion).

The following result was independently proved by Bychkov, Gorbounov, Kazakov and Talalaev [BGKT21] and Chepuri, George and Speyer [CGS21], following earlier results of Lam [Lam18].

Theorem 3.7. *The composition* $\mathrm{Meas}_{G_+} \circ j_G^+ : \mathcal{R}_G \xrightarrow{\sim} \mathrm{IG}^{\Omega}_{>0}(n+1,2n)$ *is a homeomorphism.*

Therefore, we have a commuting diagram

$$\begin{array}{ccc} \mathcal{R}_G & \xhookrightarrow{\;j_G^+\;} & \mathcal{X}_{G_+} \\ {\scriptstyle \mathrm{Meas}_{G_+}\circ j_G^+}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle \mathrm{Meas}_{G_+}} \\ \mathrm{IG}^{\Omega}_{>0}(n+1,2n) & \hookrightarrow & \mathrm{Gr}_{>0}(n+1,2n) \end{array}.$$

3.3. **A bit of representation theory of the spin group.** In this section, we give a brief background on the spin group, mostly following [FH91, Chapter 20] and [HS10, Section 5], and prove Proposition 3.9 relating Cartan and Plücker coordinates. Consider the nondegenerate symmetric bilinear form $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ defined by

$$Q(x,y) := \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}(x_i y_{n+i} + x_{n+i} y_i).$$

We first make a change of basis so that Q becomes the standard nondegenerate symmetric bilinear form. Let W denote the Lagrangian subspace $\mathrm{span}(e_1, e_2, \ldots, e_n)$. We have an isomorphism

$$\begin{aligned} W^{\perp} &\to W^{\vee} \\ e_{n+i} &\mapsto (-1)^{i-1} e_i^{\vee}, \end{aligned} \tag{3.2}$$

where $W^{\vee}$ denotes the dual vector space of W and $e_i^{\vee}$ is basis vector dual to e_i, i.e., $e_i^{\vee}(e_j) = \delta_{ij}$. This gives rise to an isomorphism $\mathbb{C}^{2n} \cong W \oplus W^{\vee}$ such that the inner product Q becomes

$$Q((x,x^{\vee}),(y,y^{\vee})) = \frac{1}{2}(x^{\vee}(y) + y^{\vee}(x)) \text{ where } (x,x^{\vee}),(y,y^{\vee}) \in W \oplus W^{\vee}. \tag{3.3}$$

Note that our form Q agrees with [HS10] and differs from the standard form in [FH91] by a factor of $\frac{1}{2}$.

Let $\mathrm{Cl}(Q) := \bigoplus_{k=0}^{\infty}(\mathbb{C}^{2n})^{\otimes k}/\langle x \otimes x - Q(x,x)\rangle$ denote the *Clifford algebra*. Since the ideal $\langle x \otimes x - Q(x,x)\rangle$ is generated by elements of even degree, the Clifford algebra has a $\mathbb{Z}/2\mathbb{Z}$ grading: $\mathrm{Cl}(Q) = \mathrm{Cl}(Q)^{\mathrm{even}} \oplus \mathrm{Cl}(Q)^{\mathrm{odd}}$.

The *Clifford group*

$$\mathrm{Cl}^*(Q) := \{x \in \mathrm{Cl}(Q) \mid \text{there exists } y \in \mathrm{Cl}(Q) \text{ such that } x \otimes y = y \otimes x = 1\}$$

is the multiplicative group of units inside $\mathrm{Cl}(Q)$. Its Lie algebra $\mathfrak{cl}^*(Q)$ is $\mathrm{Cl}(Q)$ with the Lie bracket $[x,y] := x \otimes y - y \otimes x$, and we have the exponential map $\exp : \mathfrak{cl}^*(Q) \to \mathrm{Cl}^*(Q)$ defined by

$$\exp(x) := \sum_{n \geq 0} \frac{x^{\otimes n}}{n!}. \tag{3.4}$$

The Clifford algebra has an anti-involution $u \mapsto u^*$ called *conjugation* defined by $(x_1 \otimes \cdots \otimes x_r)^* := (-1)^r x_r \otimes \cdots \otimes x_1$. The involution $\alpha : \mathrm{Cl}(Q) \to \mathrm{Cl}(Q)$ defined by $\alpha(x_1 \otimes \cdots \otimes x_r) := (-1)^r(x_1 \otimes \cdots \otimes x_r)$ is called the *main involution*. The *pin* and *spin groups* are defined as

$$\mathrm{Pin}(Q) := \{x \in \mathrm{Cl}^*(Q) : x \otimes x^* = 1 \text{ and } \alpha(x) \otimes \mathbb{C}^{2n} \otimes x^* \subseteq \mathbb{C}^{2n}\},$$
$$\mathrm{Spin}(Q) := \{x \in \mathrm{Cl}^*(Q) \cap \mathrm{Cl}(Q)^{\mathrm{even}} : x \otimes x^* = 1 \text{ and } \alpha(x) \otimes \mathbb{C}^{2n} \otimes x^* \subseteq \mathbb{C}^{2n}\}.$$

The map $\rho : \mathrm{Pin}(Q) \to \mathrm{O}(Q)$ (resp., $\rho : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$) defined by $x \mapsto \rho(x)$ where $\rho(x) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the endomorphism $v \mapsto \alpha(x) \otimes v \otimes x^*$ makes $\mathrm{Pin}(Q)$ (resp., $\mathrm{Spin}(Q)$) a double cover of $\mathrm{O}(Q)$ (resp., $\mathrm{SO}(Q)$).

The Lie algebra of $\mathrm{SO}(Q)$ is

$$\mathfrak{so}(Q) := \{X \in \mathrm{End}(\mathbb{C}^{2n}) \mid Q(X(v), w) + Q(v, X(w)) = 0 \text{ for all } v, w \in \mathbb{C}^{2n}\}.$$

The map $\varphi : \bigwedge^2 \mathbb{C}^{2n} \to \mathfrak{so}(Q)$ sending $a \wedge b$ to $\varphi_{a\wedge b}$ given by

$$\varphi_{a\wedge b}(v) := 2(Q(b, v)a - Q(a, v)b) \tag{3.5}$$

is an isomorphism of Lie algebras. On the other hand, the map $\psi : \bigwedge^2 \mathbb{C}^{2n} \to \mathfrak{cl}^*(Q)$ sending $a \wedge b$ to $a \otimes b - Q(a, b)$ is a map of Lie algebras.

Lemma 3.8. [FH91, Lemma 20.7 and Exercise 20.33] *The composition $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \to \mathrm{Cl}(Q)^{\mathrm{even}}$ is an embedding of Lie algebras. The embedded image is the Lie algebra $\mathfrak{spin}(Q)$ of* $\mathrm{Spin}(Q)$.

Let $S := \bigwedge^\bullet W$. Define the $\mathrm{Cl}(Q)$ representation $\Gamma : \mathrm{Cl}(Q) \to \mathrm{End}(S)$ by

$$\Gamma_x(w_1 \wedge \cdots \wedge w_k) := x \wedge (w_1 \wedge \cdots \wedge w_k) \text{ for } x \in W,$$
$$\Gamma_{x^\vee}(w_1 \wedge \cdots \wedge w_k) := x^\vee \lrcorner (w_1 \wedge \cdots \wedge w_k) \text{ for } x^\vee \in W^\vee,$$

where $x^\vee \lrcorner (w_1 \wedge \cdots \wedge w_k) := \sum_{i=1}^k (-1)^{i-1} x^\vee(w_i) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k$. This is an isomorphism $\mathrm{Cl}(Q) \cong \mathrm{End}(S)$. Let $S_+ := \bigwedge^{\mathrm{even}} W$ and $S_- := \bigwedge^{\mathrm{odd}} W$. Restricting Γ, we obtain an isomorphism

$$\Gamma : \mathrm{Cl}(Q)^{\mathrm{even}} \xrightarrow{\cong} \mathrm{End}(S_+) \oplus \mathrm{End}(S_-).$$

The embedding $\mathrm{Spin}(Q) \subset \mathrm{Cl}(Q)^{\mathrm{even}}$ makes $S_\pm$ into $\mathrm{Spin}(Q)$ representations, called *half-spin representations.*

For $j \in [n]$, let $c_j(t) := t e_j \otimes e_j^\vee + t^{-1} e_j^\vee \otimes e_j$, and for $t = (t_1, \dots, t_n) \in (\mathbb{C}^\times)^n$, let

$$c(t) := \prod_{j=1}^n c_j(t_j). \tag{3.6}$$

The image of c is the *maximal torus* inside $\mathrm{Spin}(Q)$ and under the covering $\rho : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$, we get $\rho(c(t)) = \mathrm{diag}(t_1^2, \dots, t_n^2, t_1^{-2}, \dots, t_n^{-2})$ (see [FH91, Equation (23.7)]; the factor of $\frac{1}{2}$ disappears due to our convention for Q).

1 (resp., e_1) is a highest weight vector of S_+ (resp. S_-) with weight $(-1, -1, \dots, -1)$ (resp., $(1, -1, \dots, -1)$). For $I, I^\vee \subseteq [n]$ such that $\#I + \#I^\vee = n+1$, let $e_{I,I^\vee}$ denote wedge product indexed by $I \sqcup I^\vee$: If $I = \{i_1 < i_2 < \cdots < i_k\}$ and $I^\vee = \{j_1 < j_2 < \cdots < j_{n+1-k}\}$, then $e_{I,I^\vee} := e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1}^\vee \wedge \cdots \wedge e_{j_{n+1-k}}^\vee$. Since $\bigwedge^{n+1} \mathbb{C}^{2n}$ is an irreducible $\mathrm{Spin}(Q)$ representation with highest weight vector $e_{\{1\},[n]}$ with weight $(0, -2, \dots, -2)$ (where $x \in \mathrm{Spin}(Q)$ acts as the special orthogonal transformation $\rho(x)$), $\bigwedge^{n+1} \mathbb{C}^{2n}$ is a direct summand of $S_+ \otimes S_-$. Let

$p : \bigwedge^{n+1} \mathbb{C}^{2n} \hookrightarrow S_+ \otimes S_-$ denote the morphism of $\mathrm{Spin}(Q)$ representations sending $e_{1,[n]}$ to $(-1)^{\sum_{j\in[n]}(j-1)} 1 \otimes e_1$. Let $\sigma(I)$ be -1 if $\#I \equiv 2$ modulo 4 and 1 otherwise.

Proposition 3.9. *Suppose $I, I^\vee \subseteq [n]$ are such that $\#I + \#I^\vee = n+1$ and $I \cap I^\vee = \{l\}$. Then,*

$$(-1)^{\sum_{j\in I^\vee}(j-1)} p(e_{I,I^\vee}) = \sigma(I)\sigma(I\setminus\{l\}) e_I \otimes e_{I\setminus\{l\}} \quad \text{if } \#I \text{ is even, and}$$
$$(-1)^{\sum_{j\in I^\vee}(j-1)} p(e_{I,I^\vee}) = \sigma(I)\sigma(I\setminus\{l\}) e_{I\setminus\{l\}} \otimes e_I \quad \text{if } \#I \text{ is odd.} \tag{3.7}$$

Proof. We will use the action of $\mathrm{Spin}(Q)$ to send $e_{1,[n]}$ to $e_{I,I^\vee}$ and use $\mathrm{Spin}(Q)$ equivariance of p. The main difficulty will be in keeping track of the signs.

We start by defining the required elements of $\mathrm{Spin}(Q)$. By the Cartan–Dieudonné theorem [LM89, Theorem 2.7], any element of $\mathrm{O}(Q)$ can be written as a product of reflections, so we look for appropriate reflections. If $w \in V$ with $Q(w,w) = -1$ and R_w is the reflection in the hyperplane orthogonal to w, then $w \in \mathrm{Pin}(Q)$ and $\rho(w) = R_w$. Let $u_{jk} := \frac{i}{\sqrt{2}}(e_j - e_k + e_j^\vee - e_k^\vee)$ and $v_{jk} := \frac{1}{\sqrt{2}}(e_j - e_k - e_j^\vee + e_k^\vee)$ so that $Q(u_{jk}, u_{jk}) = Q(v_{jk}, v_{jk}) = -1$. A computation shows that the composition $R_{v_{jk}} \circ R_{u_{jk}}$ is in $\mathrm{SO}(Q)$ and is the transformation $e_j \longleftrightarrow e_k, e_j^\vee \longleftrightarrow e_k^\vee$. Let $w_j := e_j - e_j^\vee$, so that $Q(w_j, w_j) = -1$. The composition $R_{w_j} \circ R_{w_k} \in \mathrm{SO}(Q)$ is the transformation $e_j \longleftrightarrow e_j^\vee, e_k \longleftrightarrow e_k^\vee$. The transformations $R_{v_{jk}} \circ R_{u_{jk}}$ and $R_{w_j} \circ R_{w_k}$ have lifts $v_{jk} \otimes u_{jk}$ and $w_j \otimes w_k$ to $\mathrm{Spin}(Q)$ respectively.

Now, we proceed by induction on $m := \#I$. When $m = 1$, we have $I = \{l\}$ and $I^\vee = [n]$ for some $l \in [n]$. Suppose $l \neq 1$. Since $v_{1l} \otimes u_{1l}$ acts on $\bigwedge^{n+1} \mathbb{C}^{2n}$ as the special orthogonal transformation $\rho(v_{1l} \otimes u_{1l}) = R_{v_{1l}} \circ R_{u_{1l}}$, we have

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot e_{1,[n]} &= R_{v_{1l}} \circ R_{u_{1l}}(e_{1,[n]}) \\ &= e_l \wedge e_l^\vee \wedge e_2^\vee \wedge \cdots \wedge e_{l-1}^\vee \wedge e_1^\vee \wedge e_{l+1}^\vee \wedge \cdots \wedge e_n^\vee \\ &= -e_{l,[n]}, \end{aligned}$$

where the -1 arises when we reorder the alternating tensor. Next, we compute the action on $S_+ \otimes S_-$. We have

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot 1 &= \Gamma_{v_{1l}} \circ \Gamma_{u_{1l}}(1) \\ &= \Gamma_{v_{1l}} \circ \frac{i}{\sqrt{2}}(\Gamma_{e_1} - \Gamma_{e_l} + \Gamma_{e_1^\vee} - \Gamma_{e_l^\vee})(1) \\ &= \frac{i}{\sqrt{2}} \Gamma_{v_{1l}}(e_1 \wedge 1 - e_l \wedge 1 + e_1^\vee(1) - e_l^\vee(1)) \\ &= \frac{i}{\sqrt{2}} \Gamma_{v_{1l}}(e_1 - e_l) \\ &= \frac{i}{2}(\Gamma_{e_1} - \Gamma_{e_l} - \Gamma_{e_1^\vee} + \Gamma_{e_l^\vee})(e_1 - e_l) \\ &= \frac{i}{2}(e_1 \wedge e_1 - e_1 \wedge e_l - e_l \wedge e_1 + e_l \wedge e_l - e_1^\vee(e_1) + e_1^\vee(e_l) + e_l^\vee(e_1) - e_l^\vee(e_l)) \\ &= -i1, \end{aligned}$$

and similarly,

$$\begin{aligned} v_{1l} \otimes u_{1l} \cdot e_1 &= \Gamma_{v_{1l}} \left(\frac{i}{\sqrt{2}}(1 - e_l \wedge e_1) \right) \\ &= -ie_l. \end{aligned}$$

Therefore, $v_{1l} \otimes u_{1l} \cdot (1 \otimes e_1) = -1 \otimes e_l$. By $\mathrm{Spin}(Q)$ equivariance of p, for all $l \in [n]$, we have

$$\begin{aligned} p(e_{l,[n]}) &= p(-v_{1l} \otimes u_{1l} \cdot e_{1,[n]}) \\ &= -v_{1l} \otimes u_{1l} \cdot p(e_{1,[n]}) \\ &= -v_{1l} \otimes u_{1l} \cdot (-1)^{\sum_{j \in [n]}(j-1)} 1 \otimes e_1 \\ &= (-1)^{\sum_{j \in [n]}(j-1)} 1 \otimes e_l. \end{aligned}$$

Since $\sigma(I \setminus \{l\}) = \sigma(\varnothing) = 1$ and $\sigma(I) = \sigma(\{l\}) = 1$, we get (3.7) for this case.

Now suppose $m = \#I > 1$. Let k be the largest element of $I \setminus \{l\}$. Define $I_0 := I \setminus \{k\}$ and $I_0^\vee := I \cup \{k\}$ so that we have $\#I_0 + \#I_0^\vee = n + 1$ and $I_0 \cap I_0^\vee = \{l\}$. By a careful computation, we obtain

$$\begin{aligned} R_{w_k} \otimes R_{w_l}(e_{I_0, I_0^\vee}) &= (-1)^{(k-1)+m} e_{I, I^\vee}, \\ w_k \otimes w_l \cdot e_{I_0 \setminus \{l\}} &= (-1)^{m-1+\#\{j \in I_0 | j < l\}} e_I, \\ w_k \otimes w_l \cdot e_{I_0} &= (-1)^{m-2+\#\{j \in I_0 | j < l\}} e_{I \setminus \{l\}}. \end{aligned} \tag{3.8}$$

Since $I_0^\vee = I^\vee \cup \{k\}$, we have

$$(-1)^{\sum_{j \in I_0^\vee}(j-1)} = (-1)^{\sum_{j \in I^\vee}(j-1) + (k-1)}. \tag{3.9}$$

Since $\#(I_0 \setminus \{l\}) = \#I - 2$, we have

$$\{\#(I_0 \setminus \{l\}) \text{ modulo } 4, \#I \text{ modulo } 4\} = \begin{cases} \{0, 2\} \text{ if } m \text{ is even, and} \\ \{1, 3\} \text{ if } m \text{ is odd,} \end{cases}$$

which implies that $\sigma(I_0 \setminus \{l\})\sigma(I) = (-1)^{m+1}$. Using this and $\#I_0 = \#(I \setminus \{l\})$, we get

$$\sigma(I_0)\sigma(I_0 \setminus \{l\})\sigma(I)\sigma(I \setminus \{l\}) = \sigma(I_0 \setminus \{l\})\sigma(I) = (-1)^{m+1}. \tag{3.10}$$

Assume $m = \#I$ is even so that $\#I_0$ is odd. By the induction hypothesis,

$$(-1)^{\sum_{j \in I_0^\vee}(j-1)} p(e_{I_0, I_0^\vee}) = \sigma(I_0)\sigma(I_0 \setminus \{l\}) e_{I_0 \setminus \{l\}} \otimes e_{I_0}.$$

By (3.8) and $\mathrm{Spin}(Q)$ equivariance of p,

$$(-1)^{\sum_{j \in I_0^\vee}(j-1)} p((-1)^{(k-1)+m} e_{I, I^\vee}) = -\sigma(I_0)\sigma(I_0 \setminus \{l\}) e_I \otimes e_{I \setminus \{l\}}.$$

Using (3.9) and (3.10), we get $(-1)^{\sum_{j \in I^\vee}(j-1)} p(e_{I, I^\vee}) = \sigma(I)\sigma(I \setminus \{l\}) e_I \otimes e_{I \setminus \{l\}}$. The case when $\#I$ is odd is almost identical. □

3.4. **The positive decorated orthogonal Grassmannian.** In this section, we define the orthogonal Grassmannian and its Cartan embedding; for further background, see [Che97, BHH21, HS10].

For a subspace U of V, let $U^\perp := \{x \in V \mid Q(x, y) = 0 \text{ for every } y \in U\}$ denote its orthogonal complement. A subspace U is said to be *isotropic* (resp., *coisotropic*) for Q if $U \subseteq U^\perp$ (resp., $U^\perp \subseteq U$). Let $\mathrm{OG}(n, 2n)$ denote the orthogonal Grassmannian of isotropic n dimensional subspaces. Then $\mathrm{OG}(n, 2n) = \mathrm{OG}_+(n, 2n) \sqcup \mathrm{OG}_-(n, 2n)$ has two irreducible components, where $\mathrm{OG}_+(n,2n)$ (resp., $\mathrm{OG}_-(n,2n)$) is the $\mathrm{Spin}(Q)$ orbit of $\mathrm{span}(e_{n+1},\dots,e_{2n})$ (resp., $\mathrm{span}(e_1,e_{n+2},e_{n+3},\dots,e_{2n})$). We have $\mathrm{Spin}(Q)$ equivariant embeddings $\mathrm{Ca}_\pm : \mathrm{OG}_\pm(n,2n) \hookrightarrow \mathbb{P}(S_\pm)$, called *Cartan embeddings*, defined by

$$\mathrm{span}(e_{n+1},\dots,e_{2n}) \mapsto [1] \text{ and } \mathrm{span}(e_1,e_{n+2},e_{n+3},\dots,e_{2n}) \mapsto [e_1] \text{ respectively,}$$

where as usual, $[x]$ denotes the projectivization of x. Let $\mathrm{OG}(n+1,2n)$ denote the orthogonal Grassmannian of coisotropic $(n+1)$-dimensional subspaces. Given $X \in \mathrm{OG}(n+1,2n)$, there are two maximal isotropic subspaces $X_\pm \in \mathrm{OG}_\pm(n,2n)$ contained in X. The composition

$$\begin{array}{ccccc} \mathrm{OG}(n+1,2n) & \hookrightarrow & \mathrm{OG}_+(n,2n)\times\mathrm{OG}_-(n,2n) & \xrightarrow{\mathrm{Ca}_+\times\mathrm{Ca}_-} & \mathbb{P}(S_+)\times\mathbb{P}(S_-) \\ \uplus & & \uplus & & \uplus \\ X & \longmapsto & (X_+,X_-) & \longmapsto & (\mathrm{Ca}_+(X_+),\mathrm{Ca}_-(X_-)) \end{array}$$

defines a $\mathrm{Spin}(Q)$ equivariant embedding $\mathrm{Ca} : \mathrm{OG}(n+1,2n) \hookrightarrow \mathbb{P}(S_+)\times\mathbb{P}(S_-)$. Let

$$\widetilde{\mathrm{OG}}(n+1,2n) := \{(X,s_+,s_-) \mid X \in \mathrm{OG}(n+1,2n), s_\pm \in \mathrm{Ca}_\pm(X_\pm)\}$$

denote the *decorated orthogonal Grassmannian*. Then, we have an embedding $\widetilde{\mathrm{OG}}(n+1,2n) \hookrightarrow S_+\times S_-$ sending (X,s_+,s_-) to (s_+,s_-).

Recall that $\sigma(I)$ is defined to be -1 if $\#I \equiv 2$ modulo 4 and 1 otherwise. The coefficients $\Sigma_I(X,s_+,s_-)$ of $\sigma(I)e_I$ in (s_+,s_-) are called *Cartan coordinates*. Consider the bihomogeneous equations

$$\Sigma_{I\cup\{j,l\}}\Sigma_{I\cup\{k\}} = \Sigma_I\Sigma_{I\cup\{j,k,l\}} + \Sigma_{I\cup\{j,k\}}\Sigma_{I\cup\{l\}} + \Sigma_{I\cup\{k,l\}}\Sigma_{I\cup\{j\}}, \tag{3.11}$$

for $j<k<l$.

Theorem 3.10 (Henriques and Speyer, [HS10, Theorem 5.3])**.** *The image of $\widetilde{\mathrm{OG}}(n+1,2n)$ in $S_+\times S_-$ is the subvariety cut out by all the equations (3.11).*

Remark 3.11. The actual statement of [HS10, Theorem 5.3] is that the image of $\mathrm{OG}(n+1,2n)$ in $\mathbb{P}(S_+)\times\mathbb{P}(S_-)$ is the closed subvariety defined by the bihomogeneous equations (3.11), but this implies Theorem 3.10 because $(s_+,s_-)\neq(0,0)$ is in the image of $\widetilde{\mathrm{OG}}(n+1,2n)$ in $S_+\times S_-$ if and only if $([s_+],[s_-])$ is in the image of $\mathrm{OG}(n+1,2n)$ in $\mathbb{P}(S_+)\times\mathbb{P}(S_-)$.

Consider the $\mathrm{Spin}(Q)$ equivariant map $\eta : \widetilde{\mathrm{OG}}(n+1,2n) \to \widetilde{\mathrm{Gr}}(n+1,2n)$ defined by $(\mathrm{span}(e_1,e_1^\vee,\dots,e_n^\vee),1,e_1) \mapsto (\mathrm{span}(e_1,e_1^\vee,\dots,e_n^\vee),e_{1,[n]})$.

Remark 3.12. The maps $\eta : \widetilde{\mathrm{OG}}(n+1,2n) \to \widetilde{\mathrm{Gr}}(n+1,2n)$ and $S_+\times S_- \to S_+\otimes S_-$ are not embeddings but they become embeddings upon projectivization.

Given $I \in \binom{[2n]}{n+1}$, let

$$J := I\cap[n] \text{ and } J^\vee := \{i-n \mid i \in I\cap[n+1,2n]\}. \tag{3.12}$$

Under the change of basis (3.2), e_I becomes $(-1)^{\sum_{j\in J^\vee}(j-1)}e_{J,J^\vee}$. The following proposition relates Plücker and Cartan coordinates.

Proposition 3.13. *Let $(X,s_+,s_-) \in \widetilde{\mathrm{OG}}(n+1,2n)$, let $(X,x)=\eta(X,s_+,s_-)$, and let $J,J^\vee$ be defined as in (3.12). If $\#(J\cap J^\vee)=1$, then*

$$\Delta_I(X,x) = \Sigma_J(X,s_+,s_-)\Sigma_{[n]\setminus J^\vee}(X,s_+,s_-).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{OG}}(n+1,2n) & \hookrightarrow & S_+ \times S_- \\ \eta\downarrow & & \downarrow \\ \widetilde{\mathrm{Gr}}(n+1,2n) \hookrightarrow \bigwedge^{n+1}\mathbb{C}^{2n} & \overset{p}{\hookrightarrow} & S_+ \otimes S_- \end{array}.$$

Let $(X,s_+,s_-) \in \widetilde{\mathrm{OG}}(n+1,2n)$ and let $(X,x) = \eta(X,s_+,s_-)$. The coefficient of $\sigma(J)\sigma([n] \setminus J^\vee)e_J \otimes e_{[n]\setminus J^\vee}$ in $s_+ \otimes s_-$ is $\Sigma_J(X,s_+,s_-)\Sigma_{[n]\setminus J^\vee}(X,s_+,s_-)$. Using Proposition 3.9, and commutativity of the diagram, we get that this coefficient is also equal to $\Delta_I(X,x)$.

□

Definition 3.14. Let $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ denote the subset of $\widetilde{\mathrm{OG}}(n+1,2n)$ where all the Cartan coordinates are positive, which we call the *positive decorated orthogonal Grassmannian*. Let $\mathrm{OG}_{>0}(n+1,2n)$ denote the *positive orthogonal Grassmannian*, the image of $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ under the projection $\widetilde{\mathrm{OG}}(n+1,2n) \to \mathrm{OG}(n+1,2n)$, or equivalently, the subset of $\mathrm{OG}(n+1,2n)$ where the ratio of any two Cartan coordinates of the same parity is positive.

Example 3.15. Given $(X,s_+,s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1,2n)$, Proposition 3.13 lets us write down a matrix whose row span is X. For example, let $n=3$ and let $(X,s_+,s_-) \in \widetilde{\mathrm{OG}}_{>0}(4,6)$ be such that $(\Sigma_J(X,s_+,s_-))_{J\subseteq[3]} = (\Sigma_J)_{J\subseteq[3]}$. Then,

$$X = \text{row span}\begin{bmatrix} \Sigma_\varnothing\Sigma_1 & \Sigma_\varnothing\Sigma_2 & \Sigma_\varnothing\Sigma_3 & 0 & 0 & 0 \\ 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} & 1 & 0 & 0 \\ 0 & -\frac{\Sigma_{12}\Sigma_2}{\Sigma_\varnothing\Sigma_1} & -\frac{\Sigma_{12}\Sigma_{23}+\Sigma_{12}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & 0 & 1 & 0 \\ 0 & \frac{\Sigma_\varnothing\Sigma_{13}+\Sigma_{12}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & \frac{\Sigma_{13}\Sigma_3}{\Sigma_\varnothing\Sigma_1} & 0 & 0 & 1 \end{bmatrix},$$

where $\Sigma_2 = \frac{\Sigma_\varnothing\Sigma_{123}+\Sigma_1\Sigma_{23}+\Sigma_{12}\Sigma_3}{\Sigma_{13}}$.

3.5. **Pfaffian formulas for Cartan coordinates.** The main result of this section is Proposition 3.18 expressing each Cartan coordinate as the Pfaffian of a certain matrix. Let $A = (a_{ij})$ be a $2n \times 2n$ skew symmetric matrix. Let $\omega_A := \sum_{1\le i<j\le 2n} a_{ij}e_i \wedge e_j$ denote the associated alternating form. The pfaffian $\mathrm{pf}(A)$ of A is defined by the formula

$$\frac{1}{n!}\omega_A^{\wedge n} = \mathrm{pf}(A)e_1 \wedge \cdots \wedge e_{2n},$$

where $\omega_A^{\wedge n}$ denotes the wedge product of n copies of ω_A. For $I \subseteq [2n]$, let A_I^I denote the principal submatrix of A with rows and columns indexed by I.

Lemma 3.16 ([Pro06, Chapter 5, Equation (3.6.3)])**.** *We have*

$$\exp(\omega_A) = \sum_{I\subseteq[2n] | \#I \text{ is even}} \mathrm{pf}(A_I^I)e_I.$$

Recall from Section 3.4 that the orthogonal Grassmannian $\mathrm{OG}(n,2n) = \mathrm{OG}_+(n,2n) \sqcup \mathrm{OG}_-(n,2n)$ is the union of two components. If $X_+ \in \mathrm{OG}_+(n,2n)$ and $\Delta_{[n+1,2n]}(X_+) \neq 0$, then in the coordinates (3.2), X_+ is the row span of a matrix of the form $\begin{bmatrix} M_+ & I_n \end{bmatrix}$, where M_+ is a skew symmetric $n \times n$ matrix. Similarly, if $X_- \in \mathrm{OG}_-(n,2n)$ is such that $\Delta_{\{1,n+2,n+3,\ldots,2n\}}(X_-) \neq 0$, then X_- is the row span of an $n \times 2n$ matrix with I_n in columns $1, n+2, n+3, \ldots, 2n$ and a matrix $\tilde{M}_-$ in columns $2,3,\ldots,n+1$ such that the matrix

M_- obtained from $\tilde{M}_-$ by cyclically rotating the columns by one step to the right is skew symmetric. For $J \subseteq [n]$, let $J\Delta\{1\}$ denote the symmetric difference, i.e.,

$$J\Delta\{1\} := \begin{cases} J \setminus \{1\} & \text{if } 1 \in J; \\ J \cup \{1\} & \text{if } 1 \notin J. \end{cases}$$

Lemma 3.17. *Let $w := e_1 - e_1^\vee \in \mathrm{Pin}(Q)$ so that $\rho(w) = R_w \in \mathrm{O}(Q)$ is given by $e_1 \longleftrightarrow e_{n+1}$. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{OG}_+(n+1,2n) & \xhookrightarrow{\mathrm{Ca}_+} & \mathbb{P}(S_+) \\ {\scriptstyle R_w}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \Gamma_w} \\ \mathrm{OG}_-(n+1,2n) & \xhookrightarrow{\mathrm{Ca}_-} & \mathbb{P}(S_-) \end{array} .$$

Proof. Let $X \in \mathrm{OG}_+(n+1,2n)$, so $X = \rho(x)\cdot \mathrm{span}(e_{n+1},\dots,e_{2n})$ for some $x \in \mathrm{Spin}(Q)$. By $\mathrm{Spin}(Q)$ equivariance of Ca_+, we have

$$\Gamma_w(\mathrm{Ca}_+(X)) = [\Gamma_w \circ \Gamma_x(1)] = [\Gamma_{w\otimes x \otimes w^*} \circ \Gamma_w(1)] = [\Gamma_{w\otimes x\otimes w^*}(e_1)].$$

On the other hand, noting that $w \otimes x \otimes w^* \in \mathrm{Spin}(Q)$ and using $\mathrm{Spin}(Q)$ equivariance of Ca_-, we get

$$\begin{aligned} \mathrm{Ca}_-(R_w \cdot X) &= \mathrm{Ca}_-(\rho(w\otimes x)\cdot \mathrm{span}(e_{n+1},\dots,e_{2n})) \\ &= \mathrm{Ca}_-(\rho(w\otimes x\otimes w^*)\rho(w)\cdot \mathrm{span}(e_{n+1},\dots,e_{2n})) \\ &= \mathrm{Ca}_-(\rho(w\otimes x\otimes w^*)\cdot \mathrm{span}(e_1, e_{n+2},\dots,e_{2n})) \\ &= [\Gamma_{w\otimes x\otimes w^*}(e_1)] \\ &= \Gamma_w(\mathrm{Ca}_+(X)). \end{aligned}$$

□

Proposition 3.18. *Let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ and let (X_+, X_-) denote the maximal isotropic subspaces in X. Let M_+, M_- be as above. Then,*

$$\Sigma_J(X,s_+,s_-) = \begin{cases} \Sigma_\varnothing(X,s_+,s_-)\,\mathrm{pf}((M_+)_J^J) & \text{if } \#J \text{ is even;} \\ (-1)^{\frac{\#J-1}{2}}\Sigma_{\{1\}}(X,s_+,s_-)\,\mathrm{pf}((M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}) & \text{if } \#J \text{ is odd.} \end{cases}$$

Proof. We use $\mathrm{Spin}(Q)$ equivariance of the Cartan map and the following commutative diagram of exponential maps:

$$\begin{array}{ccc} \mathfrak{so}(Q) & \xrightarrow[\sim]{\psi\circ\varphi^{-1}} & \mathfrak{spin}(Q) \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ \mathrm{SO}(Q) & \xleftarrow{\rho} & \mathrm{Spin}(Q) \end{array} ,$$

where the exp on the left is the matrix exponential map and on the right is (3.4).

Consider the element $m_+ := \begin{bmatrix} 0 & -M_+ \\ 0 & 0 \end{bmatrix} \in \mathfrak{so}(Q)$. Exponentiating m_+, we get $\begin{bmatrix} I_n & -M_+ \\ 0 & I_n \end{bmatrix} \in \mathrm{SO}(Q)$, so that we have $\begin{bmatrix} 0 & I_n \end{bmatrix} (\exp(m_+))^T = \begin{bmatrix} M_+ & I_n \end{bmatrix}$. On the other hand, under the isomorphism $\psi\circ\varphi^{-1} : \mathfrak{so}(Q) \xrightarrow{\sim} \mathfrak{spin}(Q)$, m_+ goes to $-\sum_{1\le i<j\le n}(M_+)_{ij} e_i \otimes e_j$. Exponentiating,

and using Lemma 3.16 along with $\mathrm{Spin}(Q)$ equivariance of the Cartan map, we get

$$[s_+] = [\exp(m_+)\cdot 1] = \left[\sum_{J \text{ even}} \mathrm{pf}((-M^+)_J^J) e_J\right]. \tag{3.13}$$

Since $\Sigma_J(X, s_+, s_-)$ is the coefficient of $\sigma(J)e_J$ in s_+, we get

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_\varnothing(X, s_+, s_-)} = \sigma(J)\,\mathrm{pf}((-M^+)_J^J) = \mathrm{pf}((M^+)_J^J),$$

where we used $\mathrm{pf}((-M^+)_J^J) = (-1)^{\frac{\#J}{2}}\,\mathrm{pf}((M^+)_J^J)$ and $(-1)^{\frac{\#J}{2}} = \sigma(J)$.

Let $w := e_1 - e_1^\vee \in \mathrm{Pin}(Q)$ as in Lemma 3.17. Since $\rho(w) = R_w \in \mathrm{O}(Q)$ is given by $e_1 \longleftrightarrow e_{n+1}$, it sends X_- to row span $\begin{bmatrix} M_- & I_n \end{bmatrix} \in \mathrm{OG}_+(n+1, 2n)$. Moreover, $\Gamma_w : S_+ \to S_-$ is given by

$$\Gamma_w(e_J) = \begin{cases} -e_{J\Delta\{1\}} & \text{if } 1 \in J; \\ e_{J\Delta\{1\}} & \text{if } 1 \notin J. \end{cases}$$

Therefore, by Lemma 3.17, we get

$$\left[-\sum_{J \text{ odd}|1\in J} \Sigma_J(X, s_+, s_-) e_{J\Delta\{1\}} + \sum_{J \text{ odd}|1\notin J} \Sigma_J(X, s_+, s_-) e_{J\Delta\{1\}}\right] = \left[\sum_{J \text{ odd}} \mathrm{pf}((-M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}) e_{J\Delta\{1\}}\right].$$

Now, we have to check two cases. If $1 \in J$, then

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_{\{1\}}(X, s_+, s_-)} = (-1)^{\frac{\#J-1}{2}}\,\mathrm{pf}((M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}),$$

and if $1 \notin J$, then

$$\frac{\Sigma_J(X, s_+, s_-)}{\Sigma_{\{1\}}(X, s_+, s_-)} = -(-1)^{\frac{\#J+1}{2}}\,\mathrm{pf}((M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}) = (-1)^{\frac{\#J-1}{2}}\,\mathrm{pf}((M_-)_{J\Delta\{1\}}^{J\Delta\{1\}}).$$

□

Example 3.19. Recall Example 3.15. After making the change of basis (3.2), the two maximal isotropic subspaces X_+ and X_- are the row spans of

$$\begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} & 1 & 0 & 0 \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 & \frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 & 1 & 0 \\ -\frac{\Sigma_{13}}{\Sigma_\varnothing} & -\frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \frac{\Sigma_2}{\Sigma_1} & \frac{\Sigma_3}{\Sigma_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Sigma_{123}}{\Sigma_1} & -\frac{\Sigma_2}{\Sigma_1} & 1 & 0 \\ 0 & \frac{\Sigma_{123}}{\Sigma_1} & 0 & -\frac{\Sigma_3}{\Sigma_1} & 0 & 1 \end{bmatrix} \text{ respectively.}$$

Therefore,

$$M_+ = \begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & \frac{\Sigma_{13}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 & \frac{\Sigma_{23}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{13}}{\Sigma_\varnothing} & -\frac{\Sigma_{23}}{\Sigma_\varnothing} & 0 \end{bmatrix} \text{ and } M_- = \begin{bmatrix} 0 & \frac{\Sigma_2}{\Sigma_1} & \frac{\Sigma_3}{\Sigma_1} \\ -\frac{\Sigma_2}{\Sigma_1} & 0 & -\frac{\Sigma_{123}}{\Sigma_1} \\ -\frac{\Sigma_3}{\Sigma_1} & \frac{\Sigma_{123}}{\Sigma_1} & 0 \end{bmatrix},$$

using which we verify Proposition 3.18. For example,

$$\Sigma_\varnothing \operatorname{pf}((M_+)_{12}^{12}) = \Sigma_\varnothing \operatorname{pf}\begin{bmatrix} 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} \\ -\frac{\Sigma_{12}}{\Sigma_\varnothing} & 0 \end{bmatrix} = \Sigma_{12} \text{ when } J = \{1,2\},$$

$$(-1)^{\frac{1-1}{2}}\Sigma_1 \operatorname{pf}((M_-)_{12}^{12}) = \Sigma_1 \operatorname{pf}\begin{bmatrix} 0 & \frac{\Sigma_2}{\Sigma_1} \\ -\frac{\Sigma_2}{\Sigma_1} & 0 \end{bmatrix} = \Sigma_2 \text{ when } J = \{2\}, \text{ and}$$

$$(-1)^{\frac{2-1}{2}}\Sigma_1 \operatorname{pf}((M_-)_{23}^{23}) = -\Sigma_1 \operatorname{pf}\begin{bmatrix} 0 & \frac{-\Sigma_{123}}{\Sigma_1} \\ \frac{\Sigma_{123}}{\Sigma_1} & 0 \end{bmatrix} = \Sigma_{123} \text{ when } J = \{1,2,3\}.$$

3.6. B **variables.** Consider the space $\mathcal{B}_G := \mathbb{R}_{>0}^{V(G)\sqcup F(G)}$ of functions $B : V(G) \sqcup F(G) \to \mathbb{R}_{>0}$. We call a pair (G, B) a *B-network*. Since there is a bijection $V(G) \sqcup F(G) \xrightarrow{\sim} W(G_+)$, we will sometimes write B_{w_u} instead of B_u for $u \in V(G) \sqcup F(G)$.

A Y-Δ move $G \rightsquigarrow G'$ induces a homeomorphism $\mathcal{B}_G \xrightarrow{\sim} \mathcal{B}_{G'}$ given by the *cube recurrence*

$$B_{f_0'} := \frac{B_{v_1}B_{f_1} + B_{v_2}B_{f_2} + B_{v_3}B_{f_3}}{B_{v_0}}$$

and $B_{v'} := B_v$ for all other $v \in V(G')$ and $B_{f'} := B_f$ for all other $f \in F(G')$, where vertices and faces are labeled as in Figure 6. Define $\mathcal{B}_n := \bigsqcup_{\tau_G=\tau_n} \mathcal{B}_G \big/ \text{moves}$.

Definition 3.20. Each face g of G_+ is incident to two white vertices w_v and w_f, where $v \in V(G)$ and $f \in F(G)$. Define the inclusion $i_G^+ : \mathcal{B}_G \hookrightarrow \mathcal{A}_{G_+}$ by $A_g := B_v B_f$.

Proposition 3.21 (Goncharov and Kenyon, [GK13, Lemma 5.11])**.** *If G and G' are related by a Y-Δ move, then there is a sequence of moves relating G_+ and G'_+ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xhookrightarrow{i_G^+} & \mathcal{A}_{G_+} \\ {\scriptstyle Y\text{-}\Delta\ move}\downarrow\wr & & \wr\downarrow{\scriptstyle moves} \\ \mathcal{B}_{G'} & \xhookrightarrow{i_{G'}^+} & \mathcal{A}_{G'_+} \end{array}$$

Therefore, the inclusions i_G^+ glue to an inclusion $i^+ : \mathcal{B}_n \hookrightarrow \mathcal{A}_{\pi_{n+1,2n}}$.

Definition 3.22. Given G a reduced graph with $\tau_G = \tau_n$, we assign to each vertex and face of G a subset of $[n]$ as follows. For $j \in [n]$, let β_j denote the strand in G between t_j and t_{n+j}. The faces of the medial graph $G^\times$ are in bijection with $V(G) \sqcup F(G)$. We say that $u \in V(G) \sqcup F(G)$ is to the left of β_j if the corresponding face of $G^\times$ is to the left of β_j when β_j is oriented from t_{n+j} to t_j. For $u \in V(G) \sqcup F(G)$, define

$$J(u) := \{j \in [n] \mid u \text{ is to the left of } \beta_j\}.$$

Lemma 3.23. *Let g be a face of G_+ incident to white vertices $\mathrm{w}_v, \mathrm{w}_f$ where $v \in V(G)$ and $f \in F(G)$. If $I := S(g)$, then $\{J, [n] \setminus J^\vee\} = \{J(v), J(f)\}$, where J and $J^\vee$ are as in (3.12).*

Proof. Let α_i denote the strand in G_+ ending at d_i^+ as in Section 3.2 so that α_i and α_{n+i} are the two strands in G_+ that correspond to the strand β_i in G. Note that the faces of G_+ are in bijection with edges of $G^\times$. Let $l \in [n]$ be such that β_l is the strand containing the edge of $G^\times$ corresponding to g. Without loss of generality, assume that v is to the left and f to the right of β_l. Then, clearly we have $J(v) = J(f) \cup \{l\}$. Note that $l, n+l \in S(g)$ since

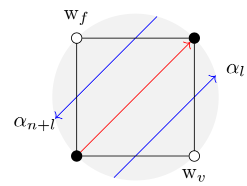


FIGURE 9. The strands of G and G_+ near a face g. The red strand is β_l oriented from t_{n+l} to t_l.

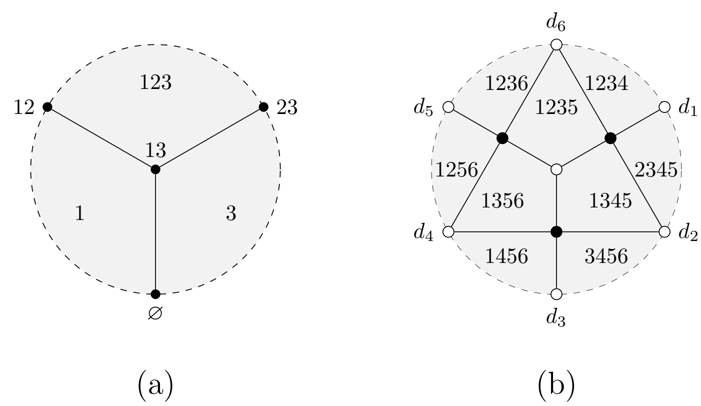


FIGURE 10. (a) Labeling the vertices and faces of the electrical network from Figure 5(a) and (b) the face labels of G_+.

g is to the left of both α_l and α_{n+l} (Figure 9). For $i \neq l$, there is exactly one strand in the pair $\{\alpha_i, \alpha_{n+i}\}$ that g is to the left of: if $i \in J(f)$, then this is α_i and if $i \notin J(v)$, then this is α_{n+i}. Any $\beta_i, i \neq l$ is of one of these two types (i.e., $[n] \setminus J(v) \sqcup J(f) = [n] \setminus \{l\}$), so we get

$$S(g) = \{l, n+l\} \cup \{i \mid i \in J(f)\} \cup \{n+i \mid i \notin J(v)\}.$$

Therefore, $J = S(g) \cap [n] = J(f) \cup \{l\} = J(v)$ and $J^\vee = ([n] \setminus J(v)) \cup \{l\} = [n] \setminus J(f)$. $\square$

Example 3.24. Figures 10(a) and (b) show the labels J and S for G and G_+ from Figure 5.

Define the map

$$\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \to \mathcal{B}_G$$

sending (X, s_+, s_-) to $(\Sigma_{J(u)}(X, s_+, s_-))_{u \in V(G) \sqcup F(G)}$.

Theorem 3.25 (Henriques and Speyer, [HS10]). *For every reduced G with $\tau_G = \tau_n$, $\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \xrightarrow{\sim} \mathcal{B}_G$ is a homeomorphism.*

Suppose G and G' are related by a Y-Δ move with vertices and faces labeled as in Figure 6. Then, up to cyclic rotation of the tuple $(v_1, f_1, v_2, f_2, v_3, f_3)$, we have

$$\begin{aligned} &J(v_0) = I \cup \{j, l\}, && J(v_1) = I, && J(v_2) = I \cup \{j, k\}, && J(v_3) = I \cup \{k, l\}, \\ &J(f_0) = I \cup \{k\}, && J(f_1) = I \cup \{j, k, l\}, && J(f_2) = I \cup \{l\}, && J(f_3) = I \cup \{j\}. \end{aligned}$$

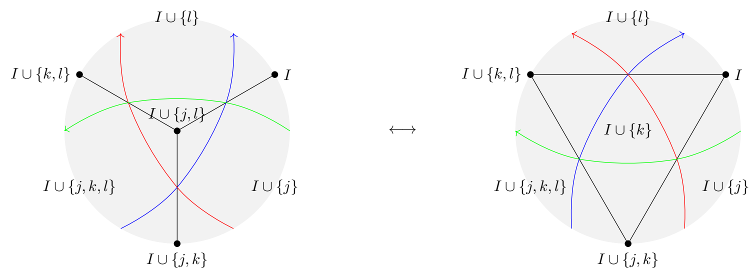


FIGURE 11. Labeling of the vertices and faces in a Y-Δ move. The green, red and blue strands are β_j, β_k and β_l respectively. The strand β_i for $i \in \{j,k,l\}$ is oriented from t_{n+i} to t_i. It is a consequence of G being reduced that locally, the orientations of the strands must be as shown here up to a cylic rotation. Indeed, since each pair of strands crosses once inside the small disk R shown in the figure where the Y-Δ move occurs, they cannot cross again in $\mathbb{D} \setminus R$. Therefore, the cyclic order of their endpoints around the boundary of R is the same as the cyclic order of their endpoints around the boundary of $\mathbb{D}$, so we have three consecutive in-endpoints followed by three consecutive out-endpoints.

for some $I \subset [n]$ and $j < k < l$ (see Figure 11). By Theorem 3.10, the following diagram commutes

$$\begin{array}{ccccc} & & \mathcal{B}_G & & \\ & \overset{\Psi_G}{\underset{\sim}{\nearrow}} & & \searrow & \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & & \sim\Big\downarrow \text{Y-}\Delta\text{ move} & & \mathcal{B}_n \\ & \underset{\Psi_{G'}}{\overset{\sim}{\searrow}} & & \nearrow & \\ & & \mathcal{B}_{G'} & & \end{array},$$

so we obtain a well-defined homeomorphism $\Psi : \widetilde{\mathrm{OG}}_{>0}(n+1,2n) \xrightarrow{\sim} \mathcal{B}_n$.

Proposition 3.26. *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\quad i_G^+ \quad} & \mathcal{A}_{G_+} \\ \Psi_G \Big\uparrow \sim & & \sim \Big\uparrow \Phi_{G_+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & \longrightarrow & \widetilde{\mathrm{Gr}}_{>0}(n+1,2n) \end{array} . \tag{3.14}$$

Proof. Let $(X,x) \in \widetilde{\mathrm{OG}}(n+1,2n)$ and let $(X,s_+,s_-) = \eta(X,x)$. Let g be a face of G_+ incident to white vertices $\mathrm{w}_v, \mathrm{w}_f$ where $v \in V(G)$ and $f \in F(G)$. Using Lemma 3.23 and Proposition 3.13, we get

$$\Delta_{S(g)}(X,x) = \Sigma_{J(v)}(X,s_+,s_-)\Sigma_{J(f)}(X,s_+,s_-).,\tag{3.15}$$

which implies that (3.14) commutes. □

4. THE ELECTRICAL RIGHT TWIST

In this section, we define the electrical right twist and prove Theorem 1.1.

Definition 4.1. Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. Following equation (56) in [GK13, Section 5.3.1], define $q_G : \mathcal{B}_G \to \mathcal{R}_G$ by $c(e) := \frac{B_u B_v}{B_f B_g}$.

Proposition 4.2. [GK13, Section 5.3.2] *If G and G' are related by a Y-Δ move, then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{q_G} & \mathcal{R}_G \\ {\scriptstyle Y\text{-}\Delta\ move}\downarrow\wr & & \wr\downarrow{\scriptstyle Y\text{-}\Delta\ move} \\ \mathcal{B}_{G'} & \xrightarrow{q_{G'}} & \mathcal{R}_{G'} \end{array}.$$

Therefore, the maps q_G glue to a map $q : \mathcal{B}_n \to \mathcal{R}_n$.

Recall the action (2.16) of $\mathbb{R}^{2n}_{>0}$ on $\mathrm{Gr}_{>0}(n+1, 2n)$ by rescaling columns.

Definition 4.3. Let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$, and let $t_i := \frac{\Sigma_{J(d_{i-1})}(X, s_+, s_-)}{\Sigma_{J(d_i)}(X, s_+, s_-)}$ for $i \in [2n]$. The *electrical right twist* of (X, s_+, s_-), denoted $\vec{\tau}_{\mathrm{elec}}(X, s_+, s_-)$, is defined to be $t \cdot \vec{\tau}(X) \in \mathrm{Gr}_{>0}(n+1, 2n)$.

Theorem 4.4. *Let G be a reduced graph with $\tau_G = \tau_n$. The image of $\vec{\tau}_{\mathrm{elec}}$ is contained in $\mathrm{IG}^{\Omega}_{>0}(n+1, 2n)$, and the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{q_G} & \mathcal{R}_G \\ \Psi_G \uparrow\wr & & \wr\downarrow \mathrm{Meas}_{G_+} \circ j^+_G \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) \end{array}, \quad \begin{array}{ccc} \mathcal{B}_n & \xrightarrow{q} & \mathcal{R}_n \\ \Psi \uparrow\wr & & \wr\downarrow \mathrm{Meas} \circ j^+ \\ \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) & \xrightarrow{\vec{\tau}_{\mathrm{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1, 2n) \end{array}.$$

Proof. We will show commutativity of the left diagram by showing that $\mathrm{Meas}^{-1}_{G_+} \circ \vec{\tau}_{\mathrm{elec}} = j^+_G \circ q_G \circ \Psi_G$. The right diagram is then obtained by gluing.

Define $B := \Psi_G(X, s_+, s_-)$, $A := i^+_G(B)$, and $t_i := \frac{B_{d_{i-1}}}{B_{d_i}}$. We have

$$\begin{aligned} \mathrm{Meas}^{-1}_{G_+} \circ \vec{\tau}_{\mathrm{elec}}(X, s_+, s_-) &= \mathrm{Meas}^{-1}_{G_+}(t \cdot \vec{\tau}(X)) && \text{(Definition 4.3)} \\ &= t \cdot \mathrm{Meas}^{-1}_{G_+} \circ \vec{\tau}(X) && \text{(Lemma 2.17)} \\ &= t \cdot p_{G_+} \circ \Phi_{G_+} \circ \overleftarrow{\tau} \circ \vec{\tau}(X) && \text{(Theorem 2.13)} \\ &= t \cdot p_{G_+} \circ \Phi_{G_+}(X) && (\overleftarrow{\tau} \circ \vec{\tau} = \mathrm{id}) \\ &= t \cdot p_{G_+}(A). && \text{(Proposition 3.26)} \end{aligned} \tag{4.1}$$

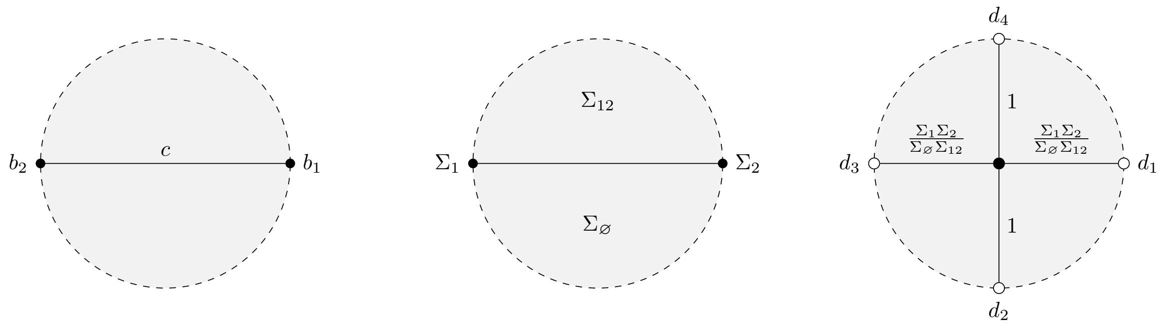
Let $e = uv$ be an edge of G and let f, g denote the faces of G incident to e. From definitions, if $[\mathrm{wt}] := j^+_G \circ q_G(B)$, then

$$\mathrm{wt}(\mathrm{b}_e \mathrm{w}_x) = \begin{cases} \frac{B_u B_v}{B_f B_g} & \text{if } x \in \{u, v\} \\ 1 & \text{if } x \in \{f, g\} \end{cases}.$$

We define a gauge transformation $\tilde{g}$ by $\tilde{g}(\mathrm{b}_e) := \frac{1}{B_u B_v}$ and

$$\tilde{g}(\mathrm{w}_u) := \begin{cases} B_u^2 & \text{if } \mathrm{w}_u \text{ is an internal white vertex, and} \\ 1 & \text{if } \mathrm{w}_u \text{ is a boundary white vertex.} \end{cases}$$

We have the following cases for edges $\mathrm{b}_e \mathrm{w}_x$ in G_+.



(a) An electrical network (G, c). (b) $\Psi_G(X, s_+, s_-)$. (c) $j_G^+ \circ q_G \circ \Psi_G(X, s_+, s_-)$.

FIGURE 12. Commutativity of the diagram in Theorem 4.4 when $n = 2$.

(1) $x = u$. Let h and h' be the two faces of G_+ incident to $\mathrm{b}_e\mathrm{w}_u$, where h is between u and f and h' is between u and g.
  (a) w_u is an internal vertex of G_+. Then, $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{1}{A_h A_{h'}} = \frac{1}{B_u^2 B_f B_g}$ to $\mathrm{b}_e\mathrm{w}_u$. Applying the gauge transformation $\tilde{g}$, we get

$$B_u B_v \frac{1}{B_u^2 B_f B_g} B_u^2 = \frac{B_u B_v}{B_f B_g}.$$

  (b) w_u is a boundary vertex d_{2i-1} of G_+. $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{B_{d_{2i-1}}}{B_{d_{2i-2}}} \frac{A_{f_{2i-1}^-}}{A_h A_{h'}} = \frac{1}{B_f B_g}$ to $\mathrm{b}_e\mathrm{w}_u$. Applying the gauge transformation $\tilde{g}$, we get $\frac{B_u B_v}{B_f B_g}$.

(2) $x = f$. Let h and h' be the two faces of G_+ incident to $\mathrm{b}_e\mathrm{w}_f$, where h is between u and f and h' is between f and v.
  (a) If w_f is an internal vertex of G_+, then $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{1}{A_h A_{h'}} = \frac{1}{B_u B_f^2 B_v}$ to $\mathrm{b}_e\mathrm{w}_f$. Applying the gauge transformation $\tilde{g}$, we get

$$B_u B_v \frac{1}{B_u B_f^2 B_v} B_f^2 = 1.$$

  (b) If w_f is the boundary vertex d_{2i}, then $\mathrm{Meas}_{G_+}^{-1} \circ \vec{\tau}_{\mathrm{elec}}$ assigns weight $\frac{B_{d_{2i}}}{B_{d_{2i-1}}} \frac{A_{f_{2i}^-}}{A_h A_{h'}} = \frac{1}{B_u B_v}$ to $\mathrm{b}_e\mathrm{w}_f$. Applying the gauge transformation $\tilde{g}$, we get 1.

Finally, $\vec{\tau}_{\mathrm{elec}}(X, s_+, s_-) = \mathrm{Meas}_{G_+} \circ j_G^+ \circ q_G \circ \Psi_G(X, s_+, s_-) \in \mathrm{IG}_{>0}^{\Omega}(n+1, 2n)$ by Theorem 3.7. □

Corollary 4.5. *The electrical right twist* $\vec{\tau}_{\mathrm{elec}} : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \to \mathrm{IG}_{>0}^{\Omega}(n+1, 2n)$ *is surjective.*

Proof. [KW17, Proposition 4] shows that q_G is surjective. By Theorem 4.4, $\vec{\tau}_{\mathrm{elec}}$ is surjective. □

Example 4.6. Let $n = 2$ and let $(X, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(3, 4)$ be such that $(\Sigma_J(X, s_+, s_-))_{J\subseteq[2]} = (\Sigma_J)_{J\subseteq[2]}$. Then, X is the row span of the matrix

$$\begin{bmatrix} \Sigma_\varnothing\Sigma_1 & \Sigma_\varnothing\Sigma_2 & 0 & 0 \\ 0 & \frac{\Sigma_{12}}{\Sigma_\varnothing} & 1 & 0 \\ 0 & -\frac{\Sigma_{12}\Sigma_2}{\Sigma_\varnothing\Sigma_1} & 0 & 1 \end{bmatrix}, \text{ so we compute } \vec{\tau}_{\mathrm{elec}}(X, s_+, s_-) = \begin{bmatrix} \frac{\Sigma_{12}}{\Sigma_\varnothing\Sigma_1\Sigma_2} & \frac{1}{\Sigma_\varnothing^2} & 0 & 0 \\ 0 & 0 & \frac{\Sigma_\varnothing}{\Sigma_1} & \frac{\Sigma_2}{\Sigma_{12}} \\ \frac{\Sigma_\varnothing}{\Sigma_2} & 0 & 0 & \frac{\Sigma_1}{\Sigma_{12}} \end{bmatrix},$$

whose Plücker coordinates are

$$\Delta_{123} = \frac{1}{\Sigma_1\Sigma_2}, \Delta_{124} = \frac{1}{\Sigma_\varnothing\Sigma_{12}}, \Delta_{134} = \frac{1}{\Sigma_1\Sigma_2}, \Delta_{234} = \frac{1}{\Sigma_\varnothing\Sigma_{12}}. \tag{4.2}$$

Consider the electrical network in Figure 12(a). Using Figure 12(c), we compute

$$\mathrm{Meas}_{G_+} \circ j_G^+ \circ q_G \circ \Psi_G(X, s_+, s_-) = \left[e_{123} + \frac{\Sigma_1\Sigma_2}{\Sigma_\varnothing\Sigma_{12}} e_{124} + e_{134} + \frac{\Sigma_1\Sigma_2}{\Sigma_\varnothing\Sigma_{12}} e_{234}\right],$$

which agrees with (4.2) upon multiplying by $\Sigma_1\Sigma_2$.

5. The electrical left twist

In this section, we define the electrical left twist and prove Theorem 1.2. By Theorem 2.9, the right twist is a homeomorphism

$$\vec{\tau} : \widetilde{\mathrm{Gr}}_{>0}(n+1, 2n)/\mathbb{R}_{>0} \cong \mathrm{Gr}_{>0}(n+1, 2n) \xrightarrow{\sim} \mathrm{Gr}_{>0}(n+1, 2n)$$

whose inverse is the left twist. We look for a similar statement for the electrical right twist. The dimension of $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ is $\binom{n+1}{2} + 1$ [HS10, Lemma 5.7], whereas the dimension of $\mathrm{IG}^\Omega(n+1, 2n)$ is $\binom{n}{2}$ (since this is the number of edges in G, hence the dimension of $\mathcal{R}_G$), so

$$\dim \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) - \dim \mathrm{IG}^\Omega_{>0}(n+1, 2n) = n + 1.$$

We will see that there is an action of $\mathbb{R}^{n+1}_{>0}$ on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ preserving $\vec{\tau}_{\mathrm{elec}}$.

We define an action of $\mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$ on $\mathcal{B}_G$ as follows. For $s \in \mathbb{R}_{>0}$ and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$,

$$((s, t) \cdot B)_v := s \left(\frac{\prod_{i\in J(v)} t_i}{\prod_{i\notin J(v)} t_i}\right) B_v. \tag{5.1}$$

Consider also the action of $\mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$ on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ defined by:

1. $\mathbb{R}_{>0}$ acts on $\widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ by $s \cdot (X, s_+, s_-) := (X, ss_+, ss_-)$.
2. Recall from (3.6) the maximal torus $(\mathbb{C}^\times)^n$ inside $\mathrm{Spin}(Q)$ which has the parameterization $c : (\mathbb{C}^\times)^n \to \mathrm{Spin}(Q)$. Restricting to $\mathbb{R}^n_{>0} \subset (\mathbb{C}^\times)^n$, we get a copy of $\mathbb{R}^n_{>0}$ inside $\mathrm{Spin}(Q)$ parameterized by $c : \mathbb{R}^n_{>0} \to \mathrm{Spin}(Q)$. We have the action $t \cdot (X, s_+, s_-) = (X\rho(c(t))^T, c(t)s_+, c(t)s_-)$, where $\rho(c(t)) \in \mathrm{SO}(Q)$ is $\mathrm{diag}(t_1^2, \ldots, t_n^2, t_1^{-2}, \ldots, t_n^{-2})$.

Lemma 5.1. *The map* $\Psi_G : \widetilde{\mathrm{OG}}_{>0}(n+1, 2n) \to \mathcal{B}_G$ *is* $\mathbb{R}_{>0} \times \mathbb{R}^n_{>0}$ *equivariant.*

Proof. This follows from the observation that

$$c_i(t_i) \cdot e_I = \begin{cases} t_i e_I & \text{if } i \in I, \text{ and} \\ t_i^{-1} e_I & \text{if } i \notin I. \end{cases}$$

□

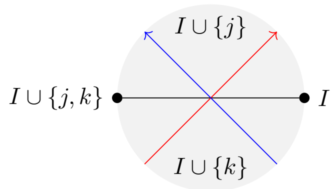


FIGURE 13. Labeling of the vertices and faces around an edge of G. The red and blue strands are β_j and β_k respectively. The strand $\beta_i, i \in \{j,k\}$, is oriented from t_{n+i} to t_i. Locally any of the four choices of orientations of the two strands is possible; however, the other three cases are cyclic rotations of the one shown.

Lemma 5.2. *The map q_G is invariant under the action (5.1).*

Proof. Let $e = uv$ be an edge of G with incident faces f, g. The map q_G assigns to e the conductance $\frac{B_u B_v}{B_f B_g}$. The four labels $(J(u), J(f), J(v), J(g))$ are some cyclic rotation of $(I, I \cup \{j\}, I \cup \{j,k\}, I \cup \{k\})$, (see Figure 13) so the factors coming from the action of (s,t) in the numerator and denominator cancel. □

By Theorem 4.4, Lemma 5.1 and Lemma 5.2, q_G and $\vec{\tau}_{\text{elec}}$ descend to the quotients to yield the commuting diagram

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}_{>0}^{n+1} & \xrightarrow{\quad q_G \quad} & \mathcal{R}_G \\ \Psi_G \big\uparrow \wr & & \wr \big\downarrow \operatorname{Meas}_{G_+} \circ j_G^+ \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}_{>0}^{n+1} & \xrightarrow{\vec{\tau}_{\text{elec}}} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array},$$

where each of the spaces has dimension $\binom{n}{2}$. We will show in Theorem 5.7 that the two horizontal maps are also homeomorphisms.

As in Section 3.2, let α_i denote the strand in G_+ from d^-_{n+i-1} to d^+_i. Let $[\mathrm{wt}](\alpha_i)$ denote the alternating product of edge weights along α_i, where the weights of edges oriented from black to white in α_i appear in the numerator and the weights of edges oriented from white to black in the denominator.

Lemma 5.3. *If* $[\mathrm{wt}] = j_G^+ \circ q_G(B)$ *and* $X = \operatorname{Meas}_{G_+}([\mathrm{wt}])$, *then* $[\mathrm{wt}](\alpha_i) = \frac{\Delta_{S(f^-_{n+i-1})}(X)}{\Delta_{S(f^-_{n+i})}(X)} = \frac{B_{d_i} B_{d_{n+i}}}{B_{d_{i-1}} B_{d_{n+i-1}}}$.

Proof. If $A := \Phi_{G_+} \circ \overleftarrow{\tau}(X)$, then by Theorem 2.13, $[\mathrm{wt}] = p_{G_+}(A)$. Let $d_{n+i-1} = \mathrm{w}_1 \xrightarrow{e_1} \mathrm{b}_1 \xrightarrow{e_2} \mathrm{w}_2 \xrightarrow{e_3} \mathrm{b}_2 \xrightarrow{e_4} \cdots \xrightarrow{e_{2k-2}} \mathrm{w}_k \xrightarrow{e_{2k-1}} \mathrm{b}_k \xrightarrow{e_{2k}} \mathrm{w}_{k+1} = d_i$ denote the sequence of vertices and edges in α_i. For each edge e_i, let g_i^- (resp. g_i^+) denote the face of G_+ on the right (resp., left) of e_i. Notice that $g^-_{2j-1} = g^-_{2j}$ for $j \in [k]$ and $g^+_{2j} = g^+_{2j+1}$ for $j \in [k-1]$. Moreover, $g_1^+ = f^-_{n+i}$

and $g_{2k}^- = f_i^-$. Therefore,

$$[\mathrm{wt}](\alpha_i) = \left(\frac{A_{g_1^+}A_{g_1^-}}{A_{f_{n+i-1}^-}}\right)\left(\frac{1}{A_{g_2^+}A_{g_2^-}}\right)\left(\frac{A_{g_3^+}A_{g_3^-}}{1}\right)\cdots\left(\frac{A_{f_i^-}}{A_{g_{2k}^+}A_{g_{2k}^-}}\right) = \frac{A_{f_{n+i}^-}}{A_{f_{n+i-1}^-}}. \tag{5.2}$$

Using Proposition 2.18(1), we get $[\mathrm{wt}](\alpha_i) = \frac{\Delta_{S(f_{n+i-1}^-)}(X)}{\Delta_{S(f_{n+i}^-)}(X)}$.

Let $[\mathrm{wt}'] := p_{G_+} \circ i_G^+(B)$. Using (5.2) for $[\mathrm{wt}']$, we get $[\mathrm{wt}'](\alpha_i) = \frac{i_G^+(B)_{f_{n+i}^-}}{i_G^+(B)_{f_{n+i-1}^-}} = \frac{B_{d_{n+i}}}{B_{d_{n+i-2}}}$. By (4.1), we have $[\mathrm{wt}] = t \cdot [\mathrm{wt}']$, where $t \in \mathbb{R}_{>0}^{2n}$ is given by $t_j = \frac{B_{d_{j-1}}}{B_{d_j}}$ for all $j \in [2n]$. Therefore, $\mathrm{wt}(e_1) = t_{n+i-1}\mathrm{wt}'(e_1)$ and $\mathrm{wt}(e_{2k}) = t_i \mathrm{wt}'(e_{2k})$, so $[\mathrm{wt}](\alpha_i) = \frac{t_i}{t_{n+i-1}}[\mathrm{wt}'](\alpha_i) = \frac{B_{d_i}B_{d_{n+i}}}{B_{d_{i-1}}B_{d_{n+i-1}}}$. □

Lemma 5.4. *Given $X \in \mathrm{IG}_{>0}^{\Omega}(n+1, 2n)$, let $t \in \mathbb{R}_{>0}^{2n}$ be such that $t_i t_{n+i} = \frac{\Delta_{S(f_{n+i}^-)}(X)}{\Delta_{S(f_{n+i-1}^-)}(X)}$. Then, $t \cdot \overleftarrow{\tau}(X) \in \mathrm{OG}_{>0}(n+1, 2n)$.*

Proof. By Corollary 4.5, there exists $(Y, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1, 2n)$ such that $\vec{\tau}_{\mathrm{elec}}(Y, s_+, s_-) = X$. If $B := \Psi_G(Y, s_+, s_-)$, then by Lemma 5.3, $t_i t_{n+i} = \frac{B_{d_{i-1}}B_{d_{n+i-1}}}{B_{d_i}B_{d_{n+i}}}$. Therefore, there exists $\lambda \in \mathbb{R}_{>0}^{2n}$ such that $t_i = \lambda_i \frac{B_{d_{i-1}}}{B_{d_i}}$ and $\lambda_{i+n} = \frac{1}{\lambda_i}$. Let $\mu \in \mathbb{R}_{>0}^{2n}$ be given by $\mu_i := \frac{B_{d_{i-1}}}{B_{d_i}}$. By definition, $\vec{\tau}_{\mathrm{elec}}(Y, s_+, s_-) = \mu \cdot \vec{\tau}(Y)$, so by Proposition 2.18(2) and Theorem 2.13, we have

$$t \cdot \overleftarrow{\tau}(X) = t \cdot \overleftarrow{\tau}(\mu \cdot \vec{\tau}(Y)) = t \cdot \mu^{-1} \cdot \overleftarrow{\tau}(\vec{\tau}(Y)) = \lambda \cdot Y.$$

Since $Y \in \mathrm{OG}_{>0}(n+1, 2n)$ and λ preserves Q, $\lambda \cdot Y \in \mathrm{OG}_{>0}(n+1, 2n)$. □

Example 5.5. Consider the electrical network (G, c) in Figure 12(a). We compute

$$\mathrm{Meas}_{G_+} \circ j_G^+(c) = [e_{123} + ce_{124} + e_{134} + ce_{234}] \in \mathrm{IG}_{>0}^{\Omega}(3, 4),$$

which is $\mathrm{Pl}(X)$ for $X = \text{row span} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & c \\ 0 & 0 & -1 & -c \end{bmatrix}$. We have

$$S(f_1^-) = 123, S(f_2^-) = 234, S(f_3^-) = 134 \text{ and } S(f_4^-) = 124,$$

so we need to choose $t \in \mathbb{R}_{>0}^4$ such that

$$t_1 t_3 = \frac{\Delta_{134}(X)}{\Delta_{234}(X)} = \frac{1}{c} \text{ and } t_2 t_4 = \frac{\Delta_{124}(X)}{\Delta_{134}(X)} = c,$$

so $t_1 = \frac{1}{ct_3}$ and $t_2 = \frac{c}{t_4}$. Then, we compute

$$t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} \frac{1}{t_3} & \frac{c}{t_4} & 0 & 0 \\ \frac{1}{t_3 c} & 0 & 0 & \frac{t_4}{c} \\ 0 & -\frac{1}{t_4} & -t_3 & 0 \end{bmatrix}.$$

To check that $t \cdot \overleftarrow{\tau}(X) \in \mathrm{OG}_{>0}(3, 4)$, we compute the orthogonal complement $(t \cdot \overleftarrow{\tau}(X))^{\perp} = \mathrm{span}(v)$, where $v = \left(\frac{1}{t_3 t_4}, \frac{c}{t_4^2}, \frac{ct_3}{t_4}, 1\right)$, and check that $Q(v, v) = \frac{1}{t_3 t_4} \cdot \frac{ct_3}{t_4} - \frac{c}{t_4^2} \cdot 1 = 0$.

Definition 5.6. Given $X \in \mathrm{IG}^{\Omega}_{>0}(n+1,2n)$, let $t \in \mathbb{R}^{2n}_{>0}$ be such that $t_i t_{n+i} = \frac{\Delta_{S(f^-_{n+i})}(X)}{\Delta_{S(f^-_{n+i-1})}(X)}$ and $t_{n+1} = 1$, and let $Y := t \cdot \overleftarrow{\tau}(X)$. By Lemma 5.4, $Y \in \mathrm{OG}_{>0}(n+1,2n)$. Let (Y, s_+, s_-) be the lift of Y to $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ such that $\Sigma_{\varnothing}(Y, s_+, s_-) = \Sigma_{\{1\}}(Y, s_+, s_-) = 1$. The *electrical left twist* $\overleftarrow{\tau}_{\mathrm{elec}} : \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \to \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0}$ is defined as $\overleftarrow{\tau}_{\mathrm{elec}}(X) := (Y, s_+, s_-)$.

Theorem 5.7. *The electrical left twist is well-defined in the sense that it is independent of the choice of $t \in \mathbb{R}^{2n}_{>0}$. The electrical right and left twists are mutually inverse homeomorphisms between $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0}$ and $\mathrm{IG}^{\Omega}_{>0}(n+1,2n)$ sitting in the commuting diagram*

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{q_G} & \mathcal{R}_G \\ \Psi_G \uparrow \wr & & \wr \downarrow \mathrm{Meas}_{G_+} \circ j^+_G \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0} & \underset{\overrightarrow{\tau}_{\mathrm{elec}},\ \sim}{\overset{\overleftarrow{\tau}_{\mathrm{elec}},\ \sim}{\rightleftarrows}} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array},$$

gluing which we get

$$\begin{array}{ccc} \mathcal{B}_n/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{q} & \mathcal{R}_n \\ \Psi \uparrow \wr & & \wr \downarrow \mathrm{Meas} \circ j^+ \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0} & \underset{\overrightarrow{\tau}_{\mathrm{elec}},\ \sim}{\overset{\overleftarrow{\tau}_{\mathrm{elec}},\ \sim}{\rightleftarrows}} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array}.$$

Proof. If $t' \in \mathbb{R}^{2n}_{>0}$ is another choice, define $\lambda \in \mathbb{R}^{2n}_{>0}$ by $\lambda_i := \frac{t'_i}{t_i}$ for all $i \in [2n]$. Note that $\lambda_1 = \lambda_{n+1} = 1$.

Let $Y' := t' \cdot \overleftarrow{\tau}(X) = \lambda \cdot Y$ and let (Y', s'_+, s'_-) denote its lift to $\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$ such that $\Sigma_{\varnothing}(Y', s'_+, s'_-) = \Sigma_{\{1\}}(Y', s'_+, s'_-) = 1$. By Lemma 5.1,

$$(\sqrt{\lambda_2 \cdots \lambda_n}, (1, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})) \cdot (Y, s_+, s_-) = (Y', s'_+, s'_-),$$

so (Y, s_+, s_-) and (Y', s'_+, s'_-) are in the same $\mathbb{R}^{n+1}_{>0}$ orbit. Therefore, $\overleftarrow{\tau}_{\mathrm{elec}}(X)$ is well-defined.

Given $(Y, s_+, s_-) \in \widetilde{\mathrm{OG}}_{>0}(n+1,2n)$, we can use the action of $\mathbb{R}^{n+1}_{>0}$ to make $\Sigma_{\varnothing}(Y, s_+, s_-) = \Sigma_{\{1\}}(Y, s_+, s_-) = 1$; indeed, we act by

$$\left(\frac{1}{\sqrt{\Sigma_{\varnothing}(Y, s_+, s_-)\Sigma_{\{1\}}(Y, s_+, s_-)}}, \left(\sqrt{\frac{\Sigma_{\varnothing}(Y, s_+, s_-)}{\Sigma_{\{1\}}(Y, s_+, s_-)}}, 1, \dots, 1 \right) \right) \in \mathbb{R}^{n+1}_{>0}.$$

If we choose $t_i := \frac{\Sigma_{J(d_{i-1})}(Y, s_+, s_-)}{\Sigma_{J(d_i)}(Y, s_+, s_-)}$ to define the electrical left twist, then $\overleftarrow{\tau}_{\mathrm{elec}} \circ \overrightarrow{\tau}_{\mathrm{elec}}(Y, s_+, s_-) = (Y, s_+, s_-)$, so $\overrightarrow{\tau}_{\mathrm{elec}}$ is injective with left inverse $\overleftarrow{\tau}_{\mathrm{elec}}$. By Corollary 4.5, $\overrightarrow{\tau}_{\mathrm{elec}}$ is also surjective, so $\overleftarrow{\tau}_{\mathrm{elec}}$ is the two-sided inverse. □

Example 5.8. Recall Example 5.5 and set $t_3 = 1$. Using row operations, we can write

$$Y := t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} 1 & \frac{c}{t_4} & 0 & 0 \\ 0 & \frac{1}{t_4} & 1 & 0 \\ 0 & -\frac{c}{t_4^2} & 0 & 1 \end{bmatrix}.$$

Letting $\Sigma_\varnothing(Y, s_+, s_-) = \Sigma_1(Y, s_+, s_-) = 1$ and comparing with the matrix in Example 4.6, we see that $\Sigma_2(Y, s_+, s_-) = \frac{c}{t_4}$ and $\Sigma_{12}(Y, s_+, s_-) = \frac{1}{t_4}$. Therefore, $q_G \circ \Psi_G(Y, s_+, s_-)$ assigns to the edge the conductance

$$\frac{\Sigma_1(Y, s_+, s_-)\Sigma_2(Y, s_+, s_-)}{\Sigma_\varnothing(Y, s_+, s_-)\Sigma_{12}(Y, s_+, s_-)} = \frac{1 \cdot \frac{c}{t_4}}{1 \cdot \frac{1}{t_4}} = c,$$

verifying commutativity of the diagram in Theorem 5.7.

6. An example of the inverse map

In this section, we work out in detail the inverse map when $n = 3$. For background on electrical networks, the Laplacian and the response matrix, see [Ken12]. Let (G, c) denote the electrical network in Figure 5(a). The Laplacian is

$$\Delta = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & u \end{matrix} \\ \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -c & a+b+c \end{bmatrix} \end{array} \begin{matrix} b_1 \\ b_2 \\ b_3 \\ u \end{matrix},$$

from which the response matrix is obtained as the Schur complement
(6.1)

$$L = -\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix} \begin{bmatrix} a+b+c \end{bmatrix}^{-1} \begin{bmatrix} -a & -b & -c \end{bmatrix} = \begin{bmatrix} -\frac{a(b+c)}{a+b+c} & \frac{ab}{a+b+c} & \frac{ac}{a+b+c} \\ \frac{ab}{a+b+c} & -\frac{b(a+c)}{a+b+c} & \frac{bc}{a+b+c} \\ \frac{ac}{a+b+c} & \frac{bc}{a+b+c} & -\frac{c(a+b)}{a+b+c} \end{bmatrix}.$$

By [CGS21, Theorem 1.8], the point $X := \mathrm{Pl}^{-1} \circ \mathrm{Meas}_{G_+} \circ j_G^+(c) \in \mathrm{IG}^\Omega_{>0}(4, 6)$ is

$$\text{row span} \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & L_{12} & 0 & -L_{12} - L_{13} \\ 0 & 0 & -1 & -L_{12} - L_{23} & 0 & L_{12} \\ 0 & 0 & 0 & L_{23} & 1 & L_{13} \end{bmatrix}.$$

Using the face labels that have been computed in Figure 10(b), to define the electrical left twist, we need to choose $t \in \mathbb{R}^6_{>0}$ such that

$$t_1t_4 = \frac{\Delta_{1456}(X)}{\Delta_{3456}(X)} = \frac{L_{23}}{L_{13}}, t_2t_5 = \frac{\Delta_{1256}(X)}{\Delta_{1456}(X)} = \frac{L_{12}}{L_{23}}, t_3t_6 = \frac{\Delta_{1236}(X)}{\Delta_{1256}(X)} = \frac{L_{13}}{L_{12}} \text{ and } t_4 = 1,$$

so let us take $t_1 = \frac{L_{23}}{L_{13}}, t_2 = L_{12}, t_3 = L_{13}, t_4 = 1, t_5 = \frac{1}{L_{23}}$ and $t_6 = \frac{1}{L_{12}}$. We compute

$$Y := t \cdot \overleftarrow{\tau}(X) = \text{row span} \begin{bmatrix} \frac{L_{12}L_{13}+L_{12}L_{23}+L_{13}L_{23}}{L_{13}} & L_{12} & 0 & 0 & 0 & -\frac{1}{L_{13}} \\ \frac{L_{23}}{L_{13}} & 0 & 0 & 0 & -\frac{1}{L_{12}} & -\frac{1}{L_{12}L_{13}} \\ -1 & -1 & -L_{13} & 0 & 0 & 0 \\ 0 & 0 & L_{12} & \frac{1}{L_{23}} & \frac{1}{L_{23}} & 0 \end{bmatrix}.$$

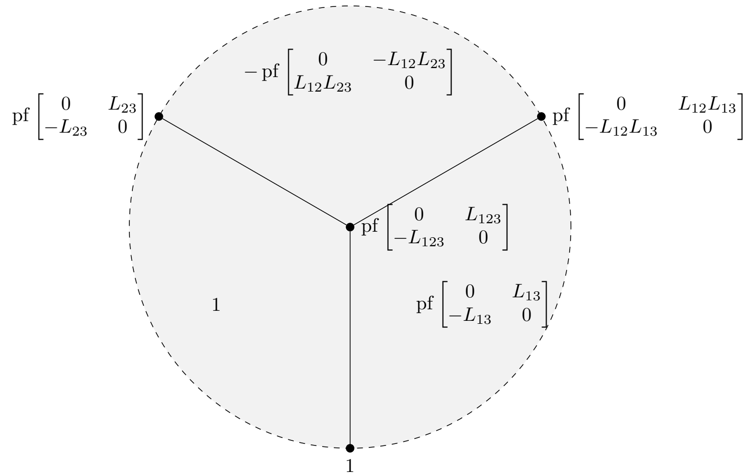


FIGURE 14. $\Psi_G \circ \overleftarrow{\tau}_{\mathrm{elec}}(X)$, where $L_{123} := L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}$.

The skew symmetric matrices M_+ and M_- as in Section 3.5 are

$$M_+ = \begin{bmatrix} 0 & L_{23} & L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} \\ -L_{23} & 0 & L_{12}L_{13} \\ -(L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}) & -L_{12}L_{13} & 0 \end{bmatrix} \text{ and}$$

$$M_- = \begin{bmatrix} 0 & 1 & L_{13} \\ -1 & 0 & -L_{12}L_{23} \\ -L_{13} & L_{12}L_{23} & 0 \end{bmatrix}.$$

Using the labels in Figure 10(a) and Proposition 3.18, we get that $\Psi_G \circ \overleftarrow{\tau}_{\mathrm{elec}}(X)$ is as shown in Figure 14, so $q_G \circ \Psi_G \circ \overleftarrow{\tau}_{\mathrm{elec}}(X)$ is given by

$$c(ub_1) = -\frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} \\ -(L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}) & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13} \\ -L_{12}L_{13} & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & L_{13} \\ -L_{13} & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & -L_{12}L_{23} \\ L_{12}L_{23} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}}{L_{23}},$$

$$c(ub_2) = \frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} \\ -(L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}) & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & L_{13} \\ -L_{13} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}}{L_{13}},$$

$$c(ub_3) = -\frac{\operatorname{pf}\begin{bmatrix} 0 & L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} \\ -(L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}) & 0 \end{bmatrix} \operatorname{pf}\begin{bmatrix} 0 & L_{23} \\ -L_{23} & 0 \end{bmatrix}}{\operatorname{pf}\begin{bmatrix} 0 & -L_{12}L_{23} \\ L_{12}L_{23} & 0 \end{bmatrix}}$$
$$= \frac{L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23}}{L_{12}}.$$

From (6.1), we have $L_{12} = \frac{ab}{a+b+c}, L_{13} = \frac{ac}{a+b+c}, L_{23} = \frac{bc}{a+b+c}$, so $L_{12}L_{13} + L_{12}L_{23} + L_{13}L_{23} = \frac{abc}{a+b+c}$. Plugging in these formulas, we get $c(ub_1) = a, c(ub_2) = b, c(ub_3) = c$.

Appendix A. Notation

In the appendix, we collect some of the notation for the spaces and maps used in the paper and the main commutative diagrams in which they sit. The third column of an entry indicates where it first appears.

A.1. **Grassmannians.**

$\mathrm{Gr}(k,n)$	Grassmannian	Section 2.1
$\widetilde{\mathrm{Gr}}(k,n)$	decorated Grassmannian	Section 2.1
$\mathrm{Gr}_{>0}(k,n)$	positive Grassmannian	Section 2.1
$\widetilde{\mathrm{Gr}}_{>0}(k,n)$	positive decorated Grassmannian	Section 2.1
$\mathrm{IG}^{\Omega}_{>0}(n+1,2n)$	positive Lagrangian Grassmannian	Section 3.2
$\mathrm{OG}_{>0}(n+1,2n)$	positive orthogonal Grassmannian	Definition 3.14
$\widetilde{\mathrm{OG}}_{>0}(n+1,2n)$	positive decorated orthogonal Grassmannian	Definition 3.14
Δ_I	Plücker coordinate	Section 2.1
Σ_J	Cartan coordinate	Section 3.4

A.2. **Bipartite graphs.**

$\mathcal{X}_\Gamma$	space of edge weights modulo gauge on Γ	Section 2.3
$\mathrm{Meas}_\Gamma : \mathcal{X}_\Gamma \to \mathrm{Gr}_{>0}(k,n)$	boundary measurement map	Section 2.3
$\mathcal{A}_\Gamma$	space parameterized by A variables	Section 2.4
$\Phi_\Gamma : \widetilde{\mathrm{Gr}}_{>0}(k,n) \to \mathcal{A}_\Gamma$	Scott's map	Section 2.4
$p_\Gamma : \mathcal{A}_\Gamma \to \mathcal{X}_\Gamma$	canonical map of cluster varieties	Definition 2.12
$\vec{\tau}$ and $\overset{\leftarrow}{\tau}$	right and left twists	Definition 2.8
	$\begin{array}{ccc} \mathcal{A}_\Gamma/\mathbb{R}_{>0} & \xrightarrow[\sim]{p_\Gamma} & \mathcal{X}_\Gamma \\ \Phi_\Gamma \uparrow \wr & & \wr \downarrow \mathrm{Meas}_\Gamma \\ \mathrm{Gr}_{>0}(k,n) & \underset{\vec{\tau}}{\overset{\overset{\leftarrow}{\tau}}{\rightleftarrows}} & \mathrm{Gr}_{>0}(k,n) \end{array}$	Theorem 2.13

A.3. **Electrical networks.**

$\mathcal{R}_G$	space of conductances on G	Section 3.2
G_+	weighted bipartite graph associated to G	Section 3.2
$j_G^+ : \mathcal{R}_G \hookrightarrow \mathcal{X}_{G_+}$	generalized Temperley's bijection	Section 3.2
$\mathcal{B}_G$	space parameterized by B variables	Section 3.6
$i_G^+ : \mathcal{B}_G \hookrightarrow \mathcal{A}_{G_+}$	A variable = product of two B variables	Definition 3.20
$\Psi_G : \widetilde{\mathrm{OG}}(n+1,2n) \to \mathcal{B}_G$	Henriques and Speyer's map	Section 3.6
$q_G : \mathcal{B}_G \to \mathcal{R}_G$	canonical map from B variables and conductances	Definition 4.1
$\vec{\tau}_{\mathrm{elec}}$	electrical right twist	Definition 4.3
$\overleftarrow{\tau}_{\mathrm{elec}}$	electrical left twist	Definition 5.6

$$\begin{array}{ccc} \mathcal{R}_G & \xhookrightarrow{\;j_G^+\;} & \mathcal{X}_{G_+} \\ {\scriptstyle \mathrm{Meas}_{G_+} \circ j_G^+}\downarrow \wr & & \wr \downarrow {\scriptstyle \mathrm{Meas}_{G_+}} \\ \mathrm{IG}^{\Omega}_{>0}(n+1,2n) & \hookrightarrow & \mathrm{Gr}_{>0}(n+1,2n) \end{array}$$

Section 3.2

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\;i_G^+\;} & \mathcal{A}_{G_+} \\ {\scriptstyle \Psi_G}\uparrow \wr & & \wr \uparrow {\scriptstyle \Phi_{G_+}} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & \longrightarrow & \widetilde{\mathrm{Gr}}_{>0}(n+1,2n) \end{array}$$

Proposition 3.26

$$\begin{array}{ccc} \mathcal{B}_G & \xrightarrow{\;q_G\;} & \mathcal{R}_G \\ {\scriptstyle \Psi_G}\uparrow \wr & & \wr \downarrow {\scriptstyle \mathrm{Meas}_{G_+} \circ j_G^+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n) & \xrightarrow{\;\vec{\tau}_{\mathrm{elec}}\;} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array}$$

Section 5

$$\begin{array}{ccc} \mathcal{B}_G/\mathbb{R}^{n+1}_{>0} & \xrightarrow[\sim]{\;q_G\;} & \mathcal{R}_G \\ {\scriptstyle \Psi_G}\uparrow \wr & & \wr \downarrow {\scriptstyle \mathrm{Meas}_{G_+} \circ j_G^+} \\ \widetilde{\mathrm{OG}}_{>0}(n+1,2n)/\mathbb{R}^{n+1}_{>0} & \underset{\vec{\tau}_{\mathrm{elec}},\ \sim}{\overset{\overleftarrow{\tau}_{\mathrm{elec}},\ \sim}{\rightleftarrows}} & \mathrm{IG}^{\Omega}_{>0}(n+1,2n) \end{array}$$

Theorem 5.7

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