

TROPICALIZATION OF FOCK'S INVERSE SPECTRAL TRANSFORM

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ABSTRACT. The dimer spectral transform gives algebro-geometric action-angle coordinates for cluster integrable systems. Fock gave an explicit formula for its inverse. We study Fock's inverse map in the tropical limit where Harnack spectral curves degenerate to nodal curves. We determine the leading order asymptotics of the classical objects appearing in Fock's formula and show that they are governed by their tropical counterparts. This yields a tropical inverse spectral transform on tropical spectral data. We show that Fock's inverse spectral transform commutes with tropicalization, and also derive a tropical version of Fay's trisecant identity.

1. INTRODUCTION

Goncharov–Kenyon [GK13] constructed a remarkable class of integrable systems from periodic bipartite dimer models called cluster integrable systems. For these systems, the spectral transform of Kenyon–Okounkov [KO07] identifies the phase space of weights on the dimer model with spectral data on a Harnack spectral curve, giving algebro-geometric action-angle coordinates. Fock [Foc15] constructed an explicit inverse to the spectral transform in terms of theta functions and abelian differentials.

In this paper, we study Fock's map in the tropical limit. Applying the spectral transform to a positive Puiseux-valued family of weights, we obtain a family of Harnack spectral curves degenerating to a nodal curve. This degeneration determines a tropical spectral curve together with tropical spectral data. Our main result is that Fock's map has a tropical limit which makes the following diagram commute:

$$\begin{array}{ccc}
 \{\text{Puiseux-valued weights}\} & \xrightarrow{\text{spectral transform}} & \{\text{family of spectral data}\} \\
 \text{tropicalization} \downarrow & & \downarrow \text{tropicalization} \\
 \{\text{tropical weights}\} & \xleftarrow{\text{tropical Fock's map}} & \{\text{tropical spectral data}\}
 \end{array}$$

Fock's map is given by an explicit formula in terms of classical objects on the spectral curve: the Abel map, the period matrix, the theta function, the vector of Riemann constants, and normalized differentials of the third kind. Our main technical contribution is to determine the leading asymptotics of these classical objects and show that they are governed by the corresponding tropical objects introduced by Mikhalkin–Zharkov [MZ08]. The Harnack property of the spectral curves plays a crucial role by ensuring that the arguments of the theta functions appearing in Fock's formula are real. This implies that the terms in the series expansions of the theta functions are positive, and so there are no cancellations among leading terms.

As a second application of our asymptotic results, we tropicalize Fay's trisecant identity. Again, the Harnack property implies that the three terms in the identity have definite signs.

Thus, the tropicalized identity takes the stronger form that a term is equal to the maximum of two other terms rather than saying that maximum of three terms is attained at least twice.

Although we consider tropical curves arising from embedded spectral curves in this paper, the underlying degeneration and asymptotic results admit a more intrinsic formulation. We may instead start with a planar abstract tropical curve and construct a corresponding family of degenerating M-curves by plumbing. Thus, many of the tropical-geometric results below are naturally statements about planar abstract tropical curves rather than embedded tropical curves arising from dimers.

Tropical limits of the dimer model on a torus have been studied by Ma in her thesis [Ma15] and by Bocklandt [Boc16a, Boc16b]. We have several motivations for studying this tropicalization. Firstly, it is a natural step toward understanding action-angle coordinates for tropical cluster integrable systems. As shown by Inoue–Lam–Pylyavskyy [ILP16], periodic box-ball systems are examples of tropical cluster integrable systems. The dynamics of such tropical integrable systems can be linearized on the Jacobian of a tropical spectral curve and expressed in terms of tropical theta functions. Tropical theta functions and tropical Fay’s trisecant identity play an important role in this theory; see, for example, Inoue–Takenawa [IT08, IT09a, IT09b], Inoue–Iwao [II12], and the review of Inoue–Kuniba–Takagi [IKT12]. Related degenerations of Riemann theta functions have also been studied by Agostini–Çelik–Struwe–Sturmfels [AcSS21], Ichikawa [Ich23] and Agostini–Fevola–Mandelstam–Sturmfels [AFMS23] in connection with tropical limits of KP solutions. In forthcoming work with Galashin [GG], we will study the tropicalization of the forward spectral transform for cluster integrable systems.

Secondly, Fock’s weights have recently found several applications in the statistical mechanics of dimers; see Boutillier–Cimasoni–de Tilière [BCdT23], Berggren–Borodin [BB25], Bobenko–Bobenko [BB26b], Boutillier–de Tilière [BdT26], and Berggren–Borodin–George [BBG26]. Tropicalization has proved to be a powerful tool in the analysis of dimer limit shapes; see Berggren–Borodin [BB26a] and Berggren–Borodin–George [BBG26]. In particular, the appendix of [BBG26] shows that suitable tropical Fock weights can be used to realize dimer models on more general bipartite graphs within the Aztec diamond.

Thirdly, in George–Galashin [GG24], we initiated the study of the boundary stratification of a compactification of the space of spectral data inspired by Postnikov’s totally nonnegative Grassmannian [Pos06]. The polyhedral structure of the positive tropical Grassmannian is closely related to the stratification of the totally nonnegative Grassmannian; see Speyer–Williams [SW05, SW21] and Arkani-Hamed–Lam–Spradlin [AHL21]. We hope that understanding the positive tropical points of spectral data will shed some light on the corresponding stratification of the compactification of the space of spectral data.

Finally, we mention a recent work of Ichikawa that is related to the present work. In [Ich26a], Fock’s dimer model is studied for families of M-curves degenerating by pinching real ovals. This degeneration is, in a sense, complementary to the one considered here.

1.1. Main results. Let G be a minimal periodic bipartite graph. In [KO07], Kenyon–Okounkov constructed the spectral transform, a map

$$\lambda : \{\text{positive edge weights on } G\} / \text{gauge} \rightarrow \{\text{spectral data associated to } G\}$$

assigning to each choice [wt] of positive edge weights modulo gauge equivalence on G a spectral datum consisting of a Harnack curve Σ , a standard divisor D , and an angle map ν assigning a point at infinity of Σ to each zig-zag path of G .

Fock [Foc15] constructed an inverse to the spectral transform which we now describe. The notation used below will be defined precisely in Section 2. At this stage, the reader should regard the formulas as displaying the structure of Fock's inverse spectral transform and its tropical analogue. The important point is that the classical objects appearing in Fock's formula are replaced, term by term, by their tropical counterparts.¹

A *wedge* in G is a pair of edges that share a black vertex. A wedge $\mathfrak{w}$ determines three zig-zag paths α, β_-, β_+ and two adjacent faces $\mathfrak{f}_-, \mathfrak{f}_+$, as shown in Figure 3. We define the wedge weight

$$(1.1) \quad \text{wt}_{\text{Fock}}(\mathfrak{w}) = -\frac{\theta(\mu_{\zeta_0}(\mathbf{d}(\mathfrak{f}_-)) - \xi + \Delta|\Pi)}{\theta(\mu_{\zeta_0}(\mathbf{d}(\mathfrak{f}_+)) - \xi + \Delta|\Pi)} \exp\left(\int_{\zeta_0}^{\alpha} \omega_{\beta_+ - \beta_-}\right).$$

Here, $\mathbf{d}$ is Fock's discrete Abel map, $\mu_{\zeta_0(t)}$ is the Abel map on $\Sigma(t)$, $\Pi(t)$ is its period matrix, θ is the Riemann theta function, $\Delta(t)$ is the vector of Riemann constants, and $\omega_{\beta_+(t) - \beta_-(t)}(t)$ is the normalized differential of the third kind with residue divisor $\beta_+(t) - \beta_-(t)$.

Any cycle γ in G can be written as a sum of wedges, $\gamma = \mathfrak{w}_1 + \cdots + \mathfrak{w}_r$. We assign to γ a cycle weight

$$(1.2) \quad [\text{wt}_{\text{Fock}}](t)(\gamma) := \pm \prod_{j=1}^r \text{wt}_{\text{Fock}}(t)(\mathfrak{w}_j).$$

These cycle weights determine a gauge-equivalence class of the edge weights $[\text{wt}_{\text{Fock}}]$, and thus we obtain a map

$$\lambda_{\text{Fock}}^{-1} : \{\text{spectral data associated to } G\} \rightarrow \{\text{positive edge weights on } G\}/\text{gauge}.$$

Fock proved that $\lambda_{\text{Fock}}^{-1} = \lambda^{-1}$.

Now let $[\text{wt}](t)$ be a positive Puiseux-valued family of edge weights modulo gauge equivalence on G , and let

$$[\text{wt}^\dagger] := \text{val}([\text{wt}](t))$$

be its tropicalization. Applying the spectral transform gives a family of spectral data

$$\lambda([\text{wt}](t)) = (\Sigma(t), D(t), \nu(t)).$$

As $t \rightarrow \infty$, the rescaled amoebas of $\Sigma(t)$ converge to an embedded tropical spectral curve $\Sigma_{\text{emb}}^\dagger$; see Figure 1. More geometrically, the family $\Sigma(t)$ together with the angle map forms a family of pointed curves degenerating to a stable nodal curve $\Sigma(\infty)$ whose irreducible components are themselves Harnack curves; see Figure 2. Such degenerations were also studied by Olarte [Ola21] in his construction of the compactification of the moduli space of Harnack curves. Associated to $\Sigma_{\text{emb}}^\dagger$ is an abstract tropical curve

$$(\Sigma^\dagger, \ell, \mathbf{g}).$$

Its underlying graph $\Sigma^\dagger$ is obtained from $\Sigma_{\text{emb}}^\dagger$ by contracting the unbounded edges and replacing edges of multiplicity greater than one by parallel edges, and records the combinatorics of the degeneration. The edge lengths ℓ record the rates at which the corresponding nodes are smoothed, while the vertex genera $\mathbf{g}$ record the genera of the irreducible components of $\Sigma(\infty)$.

¹Fock's formula is usually stated for edge weights and in terms of the prime form. We choose to state it using wedges because it allows us to express the resulting ratio of prime forms in terms of differentials of the third kind which admit a simple tropicalization. The tropicalization of the prime form appears in the physics literature in [Tou17].

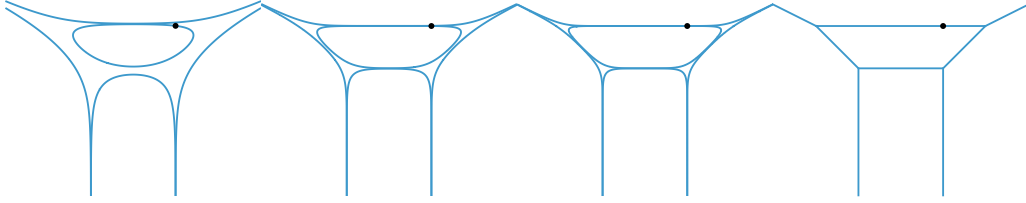


FIGURE 1. Convergence of the rescaled amoebas to the (embedded) tropical spectral curve. The panels, from left to right and top to bottom, correspond to $t = 10$, $t = 100$, $t = 1000$, and the tropical limit. The black point is the image of the standard divisor; its rescaled logarithmic image converges to the corresponding point on the tropical spectral curve.

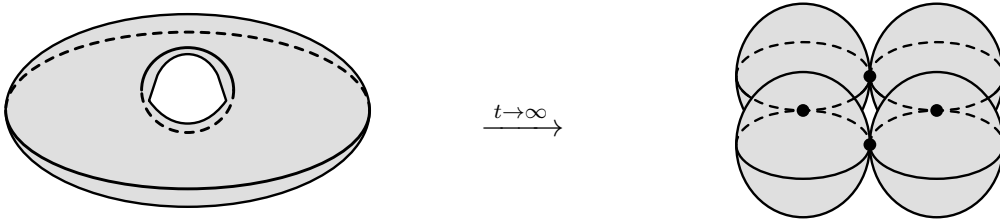


FIGURE 2. A degeneration of a smooth genus-one curve to a nodal curve consisting of four genus-zero components.

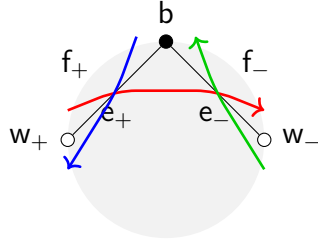


FIGURE 3. A wedge in G defined by two edges $e_- = bw_-$ and $e_+ = bw_+$ sharing a black vertex b . The zig-zag paths α , β_- , and β_+ are drawn in red, green, and blue respectively.

The standard divisor tropicalizes to an effective divisor $D^\dagger$ on $\Sigma^\dagger$ (Figure 1) and the angle map tropicalizes to a tropical angle map $\nu^\dagger : Z(G) \rightarrow V(\Sigma^\dagger)$. Thus, the family of spectral data determines a tropical spectral datum $((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger)$.

Our main technical results determine the leading asymptotics of every classical object appearing in (1.1). We obtain the asymptotics of the period matrix and Abel map using the results of Hu–Norton [HN20], and also determine the asymptotics of the normalized differentials of the third kind, the vector of Riemann constants, and the theta function. Their leading terms are governed by the corresponding tropical objects.

The Harnack property is particularly important for the theta function: the theta arguments appearing in (1.1) are real, so the terms of the theta series are positive and no cancellation can occur among terms of maximal order.

Combining these asymptotics gives a tropical version of Fock's formula:

$$\begin{aligned} \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}) &:= \theta^\dagger \left(\mu_{\zeta_0^\dagger}^\dagger(\mathbf{d}^\dagger(\mathbf{f}_-)) - \mu_{\zeta_0^\dagger}^\dagger(D^\dagger) + \Delta^\dagger \mid \Pi^\dagger \right) \\ &\quad - \theta^\dagger \left(\mu_{\zeta_0^\dagger}^\dagger(\mathbf{d}^\dagger(\mathbf{f}_+)) - \mu_{\zeta_0^\dagger}^\dagger(D^\dagger) + \Delta^\dagger \mid \Pi^\dagger \right) \\ &\quad - \int_{\zeta_0^\dagger}^{\alpha^\dagger} \omega_{\beta_+^\dagger - \beta_-^\dagger}^\dagger. \end{aligned}$$

Here and throughout, we use a $\dagger$ in the superscript to denote the tropical counterpart of a classical object. If $\gamma = \mathfrak{w}_1 + \cdots + \mathfrak{w}_r$ is a cycle, set

$$(1.3) \quad [\text{wt}_{\text{Fock}}^\dagger](\gamma) := \sum_{j=1}^r \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}_j).$$

These cycle weights determine a tropical gauge-equivalence class of edge weights. Thus, we obtain a map

$$(\lambda_{\text{Fock}}^{-1})^\dagger : \{\text{tropical spectral data associated to } G\} \rightarrow \{\text{tropical edge weights on } G\}/\text{gauge}.$$

Our first main result is:

Theorem 1.1 (cf. Theorem 6.1). *Let $[\text{wt}](t)$ be $\mathbb{K}_{>0}$ -valued edge weights modulo gauge equivalence satisfying a genericity condition (Assumption 4.1), and let $\lambda([\text{wt}](t)) = (\Sigma(t), D(t), \nu(t))$. Let $((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger)$ be the corresponding tropical spectral datum. Then,*

$$(\lambda_{\text{Fock}}^{-1})^\dagger((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger) = [\text{wt}^\dagger].$$

In other words, tropicalization commutes with Fock's inverse spectral transform.

Our second main result is a tropicalization of Fay's trisecant identity [Fay73].

Theorem 1.2 (cf. Theorem 7.2). *Let $\zeta_1^\dagger, \zeta_3^\dagger, \zeta_2^\dagger, \zeta_4^\dagger$ be four vertices in cyclic order along the outer face of $\Sigma^\dagger$. Let $\boldsymbol{\xi}^\dagger \in \mathbb{R}^{g^\dagger}$, let $\boldsymbol{\eta}_{ij}^\dagger := \mu_{\zeta_0^\dagger}^\dagger(\zeta_j^\dagger - \zeta_i^\dagger)$, and let*

$$\begin{aligned} F_1^\dagger &:= \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{13}^\dagger \mid \Pi^\dagger) + \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{24}^\dagger \mid \Pi^\dagger) - \int_{\zeta_1^\dagger}^{\zeta_3^\dagger} \omega_{\zeta_2^\dagger - \zeta_4^\dagger}^\dagger, \\ F_2^\dagger &:= \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{23}^\dagger \mid \Pi^\dagger) + \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{14}^\dagger \mid \Pi^\dagger) - \int_{\zeta_2^\dagger}^{\zeta_4^\dagger} \omega_{\zeta_1^\dagger - \zeta_3^\dagger}^\dagger, \\ F_3^\dagger &:= \theta^\dagger(\boldsymbol{\xi}^\dagger \mid \Pi^\dagger) + \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{13}^\dagger + \boldsymbol{\eta}_{24}^\dagger \mid \Pi^\dagger). \end{aligned}$$

Then,

$$F_3^\dagger = \max\{F_1^\dagger, F_2^\dagger\}.$$

Tropical versions of Fay's identity were first proved by Inoue–Takenawa [IT09a] and later generalized by Inoue–Iwao [II11] to smooth planar tropical curves. Our formulation allows general planar tropical curves. However, unlike their formulations, we restrict the points appearing in the identity to the boundary of the outer face. This restriction implies positivity of the terms appearing in Fay's identity and therefore removes the sign data appearing in the formulations of [IT09a, II11].

1.2. Organization of the paper. In Section 2, we recall the dimer spectral transform and Fock's inverse in terms of spectral data on Harnack curves. In Section 3, we associate embedded and abstract tropical spectral curves to positive Puiseux-valued edge weights and introduce the tropical differentials, Abel map, and Jacobian that will appear in the tropicalization of Fock's formula. In Section 4, we describe the degeneration of the spectral curve, first as a toric degeneration and then in plumbing coordinates, and construct the degeneration-adapted paths and symplectic basis used below. In Section 5, we determine the asymptotics of the period matrix, the Abel map, differentials of the third kind, the vector of Riemann constants, and theta function. In Section 6, we combine these results to tropicalize Fock's inverse spectral transform and prove Theorem 1.1. Finally, in Section 7, we apply the asymptotic results to obtain a tropical version of Fay's trisecant identity.

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2. DIMER SPECTRAL DATA AND FOCK WEIGHTS

2.1. Bipartite graphs on a torus. Let $G \subset \mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be a bipartite graph embedded such that every face is an open disk, and let $\tilde{G} \subset \mathbb{R}^2$ be the corresponding periodic graph in the plane.

An *edge weight* on G is a function $\text{wt} : E(G) \rightarrow \mathbb{R}_{>0}$. Two edge weights wt_1 and wt_2 are *gauge equivalent* if there exists a function

$$f : B(G) \sqcup W(G) \rightarrow \mathbb{R}_{>0}$$

such that

$$\text{wt}_2(\text{bw}) = f(\text{b})^{-1} \text{wt}_1(\text{bw}) f(\text{w}) \quad \text{for all } \text{bw} \in E(G).$$

The gauge equivalence class of wt is denoted by $[\text{wt}]$.

For every cycle $\gamma = (\text{b}_1 \xrightarrow{\text{e}_1} \text{w}_1 \xrightarrow{\text{e}_2} \text{b}_2 \xrightarrow{\text{e}_3} \dots \xrightarrow{\text{e}_{2k-2}} \text{b}_k \xrightarrow{\text{e}_{2k-1}} \text{w}_k \xrightarrow{\text{e}_{2k}} \text{b}_{k+1} = \text{b}_1)$ in G , define the *cycle weight*

$$(2.1) \quad [\text{wt}](\gamma) := \frac{\text{wt}(\text{e}_1) \cdot \text{wt}(\text{e}_3) \cdots \text{wt}(\text{e}_{2k-1})}{\text{wt}(\text{e}_2) \cdot \text{wt}(\text{e}_4) \cdots \text{wt}(\text{e}_{2k})}.$$

When $\gamma = \partial \text{f}$ is the (counterclockwise-oriented) boundary of a face $\text{f} \in F(G)$, we write

$$X_{\text{f}} := [\text{wt}](\partial \text{f})$$

for the corresponding *face weight*. The face weights $(X_{\text{f}})_{\text{f} \in F(G)}$ are the *X-cluster variables* in the sense of Fock–Goncharov [FG09], or *y-cluster variables* in the sense of Fomin–Zelevinsky [FZ02]. The face weights together with the weights ρ_z, ρ_w of two cycles γ_z, γ_w that generate the homology $H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ of the torus provide coordinates on the space of edge weights modulo gauge equivalence.

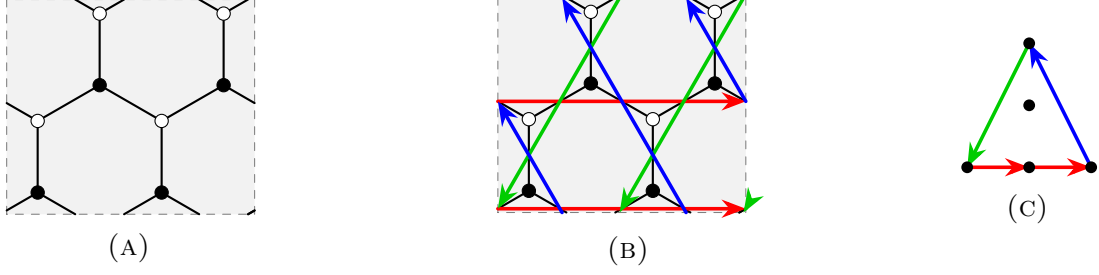


FIGURE 4. A periodic bipartite graph (A), its zig-zag paths (drawn, as is customary, as paths in the medial graph) (B), and its Newton polygon (C).

2.2. Zig-zag paths, minimality and Newton polygon. A *zig-zag path* in G or $\tilde{G}$ is an oriented path that turns maximally left at white vertices and maximally right at black vertices. We denote the set of zig-zag paths in G by $Z(G)$.

We say that G is *minimal* if, in $\tilde{G}$,

- No zig-zag path is a closed loop.
- No zig-zag path has a self-intersection.
- No two zig-zag paths have two edges $\tilde{e}_1 \neq \tilde{e}_2$ in common, with both oriented from $\tilde{e}_1$ to $\tilde{e}_2$.

Hereafter, we restrict attention to minimal graphs.

Every zig-zag path $\alpha \in Z(G)$ is a closed cycle in G and therefore determines a homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$. The unique convex lattice polygon $N \subset \mathbb{R}^2$, defined modulo translation by $\mathbb{Z}^2$, whose sides consist of the vectors $\{[\alpha] \in \mathbb{Z}^2 \mid \alpha \in Z(G)\}$ in counterclockwise cyclic order is called the *Newton polygon* of G .

For $\alpha \in Z(G)$, let $E^*(\alpha)$ denote the side of N containing $[\alpha]$. Let N° (resp. N^∂) denote the interior (resp. boundary) of N . Let $N_{\mathbb{Z}} := N \cap \mathbb{Z}^2$ (resp. $N_{\mathbb{Z}}^\circ := N^\circ \cap \mathbb{Z}^2$) denote the lattice points (resp. interior lattice points) of N .

Example 2.1. We will use as a running example the periodic bipartite graph in Figure 4(a). Its zig-zag paths are shown in Figure 4(b), and its Newton polygon in Figure 4(c).

2.3. Kasteleyn theory on the torus. Choose a fundamental rectangle for the torus with horizontal (resp. vertical) side s_z (resp. s_w). Let $e = \mathbf{b}\mathbf{w}$ be an edge of G oriented from $\mathbf{b}$ to $\mathbf{w}$. Let

$$[\mathbf{e}] := (\mathbf{e} \wedge s_w, -\mathbf{e} \wedge s_z) \in \mathbb{Z}^2,$$

where $\mathbf{e} \wedge s_w$ and $\mathbf{e} \wedge s_z$ are the signed intersection numbers of $\mathbf{e}$ with the sides of the fundamental rectangle. For $\mathbf{z} = (z, w) \in \mathbb{C}^2$ and $\mathbf{p} = (j, k) \in \mathbb{Z}^2$, we write

$$\mathbf{z}^{\mathbf{p}} = z^j w^k$$

for the corresponding monomial. Define the *Kasteleyn matrix* $\mathbf{K}(\mathbf{z}) : \mathbb{C}^{B(G)} \rightarrow \mathbb{C}^{W(G)}$ by

$$\mathbf{K}_{\mathbf{w}\mathbf{b}}(\mathbf{z}) := \sum_{e=\mathbf{b}\mathbf{w}} \text{wt}(e) \kappa(e) \mathbf{z}^{[\mathbf{e}]},$$

where the sum is over all edges with endpoints $\mathbf{b}$ and $\mathbf{w}$, and $\kappa : E(G) \rightarrow \{\pm 1\}$ is a *Kasteleyn sign*. The *characteristic polynomial* is the Laurent polynomial in z, w defined by

$$P(\mathbf{z}) := \det \mathbf{K}(\mathbf{z}).$$

Its Newton polygon agrees with Figure 4(C).

2.4. Amoebas and Harnack curves. The *amoeba* of the open spectral curve Σ° is the image

$$\mathbb{A}_{\Sigma^\circ} := \text{Log}(\Sigma^\circ), \quad \text{where} \quad \text{Log} : (\mathbb{C}^\times)^2 \rightarrow \mathbb{R}^2, \quad (z, w) \mapsto (\log |z|, \log |w|).$$

The points at infinity of Σ° correspond to *tentacles* of the amoeba. If $p \in \Sigma \cap \partial V_{E^*}^{\text{tor}}$, then the corresponding tentacle is directed along the outward-pointing normal to E^* .

Amoebas were introduced by Gelfand–Kapranov–Zelevinsky in the early 1990s [GKZ08] and have since become a central tool for connecting complex algebraic geometry with tropical and convex geometry; see for example the survey [Mik04]. Their importance for the dimer model was recognized by Kenyon–Okounkov–Sheffield [KOS06].

A real algebraic curve Σ° defined by a real Laurent polynomial $P(\mathbf{z})$ with Newton polygon N is called a (*simple*) *Harnack curve* if

$$\text{Log} : \Sigma^\circ \rightarrow \mathbb{A}_{\Sigma^\circ}$$

is at most 2–1. Log is then 2–1 over the interior of $\mathbb{A}_{\Sigma^\circ}$ and 1–1 over the boundary which coincides with $\text{Log}(\Sigma(\mathbb{R}))$. Harnack curves are classical objects; see [Mik00, MRr01] for modern developments.

Theorem 2.3 (Kenyon–Sheffield [KS04]; see also [KOS06, KO06]). *The spectral curve of a dimer model with positive edge weights is a Harnack curve.*

2.5. Abel map and Jacobian. Let Σ be a compact Riemann surface of genus g . Let

$$(\mathbf{A} = (A_1, \dots, A_g), \mathbf{B} = (B_1, \dots, B_g))$$

be a symplectic basis of $H_1(\Sigma, \mathbb{Z})$, i.e., a basis satisfying

$$(2.2) \quad A_j \wedge A_k = B_j \wedge B_k = 0, \quad A_j \wedge B_k = \delta_{jk},$$

where $\wedge$ denotes the intersection pairing. For $\mathbf{C} = (C_1, \dots, C_g)$ and $\mathbf{D} = (D_1, \dots, D_g)$ in $H_1(\Sigma, \mathbb{Z})$, we introduce the vector notation for the $g \times g$ matrix

$$\mathbf{C} \wedge \mathbf{D} = (C_j \wedge D_k)_{j,k=1}^g,$$

so that (2.2) can be written as

$$\mathbf{A} \wedge \mathbf{A} = \mathbf{B} \wedge \mathbf{B} = 0, \quad \mathbf{A} \wedge \mathbf{B} = I.$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_g)$ be the corresponding dual basis of holomorphic 1-forms on Σ , with the normalization

$$\int_{\mathbf{A}} \boldsymbol{\omega} = 2\pi i I,$$

where, for a tuple of cycles $\mathbf{C} = (C_1, \dots, C_g)$ and 1-forms $\boldsymbol{\omega} = (\omega_1, \dots, \omega_g)$, we introduce the vector notation for the $g \times g$ matrix

$$\int_{\mathbf{C}} \boldsymbol{\omega} := \left(\int_{C_j} \omega_k \right)_{j,k=1}^g.$$

The *period matrix* Π is the $g \times g$ symmetric matrix with negative-definite real part defined by

$$\Pi := \int_{\mathbf{B}} \boldsymbol{\omega}.$$

The *Jacobian variety* of Σ is the complex torus

$$\text{Jac}(\Sigma) := \mathbb{C}^g / (2\pi i \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g).$$

We denote by $[\xi]$ the image of $\xi \in \mathbb{C}^g$ in $\text{Jac}(\Sigma)$.

A *divisor* on Σ is a finite formal sum of the form

$$D = \sum_{j=1}^n c_j \zeta_j,$$

where $\zeta_j \in \Sigma$ and $c_j \in \mathbb{Z}$. The integer $\sum_{j=1}^n c_j$ is called the *degree* of D . A divisor D is said to be *effective* if $c_j \geq 0$ for all j .

The *symmetric product* of Σ of degree n is defined as

$$(2.3) \quad \text{Sym}^n(\Sigma) := \Sigma^n / \mathfrak{S}_n,$$

where the symmetric group $\mathfrak{S}_n$ acts by permuting the factors. Points of $\text{Sym}^n(\Sigma)$ correspond to effective divisors of degree n .

A *differential of the third kind* on Σ is a meromorphic 1-form on Σ with only simple poles. Given a degree-0 divisor $D = \sum_{j=1}^n c_j \zeta_j$ on Σ , there exists a unique meromorphic differential ω_D of the third kind with simple poles at $\zeta_1, \dots, \zeta_n$ and residues $c_1, \dots, c_n$, respectively, and vanishing A -periods:

$$\text{res}_{\zeta_j}(\omega_D) = c_j, \quad \text{for } 1 \leq j \leq n, \quad \int_{\mathbf{A}} \omega_D = 0;$$

see [Fay73, Chapter I]. We call ω_D the *A-normalized differential of the third kind with residue divisor D*.

Fixing a basepoint $\zeta_0 \in \Sigma$, the *Abel map*

$$\mu_{\zeta_0} : \Sigma \rightarrow \text{Jac}(\Sigma)$$

is defined by

$$\mu_{\zeta_0}(\zeta) := \int_{\zeta_0}^{\zeta} \omega = \left(\int_{\zeta_0}^{\zeta} \omega_j \right)_{j=1}^g \mod 2\pi i \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g.$$

The Abel map extends linearly to divisors.

2.6. Theta function and Riemann's theorem. The *theta function* associated to Σ is the holomorphic function

$$\theta(\cdot \mid \Pi) : \mathbb{C}^g \rightarrow \mathbb{C}, \quad \theta(\xi \mid \Pi) := \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\left(\frac{1}{2} \langle \mathbf{m}, \Pi \mathbf{m} \rangle + \langle \mathbf{m}, \xi \rangle\right).$$

Theorem 2.4 (Riemann's theorem [Mum07, Thm. 3.1]). *The Abel map*

$$\mu_{\zeta_0} : \text{Sym}^g(\Sigma) \rightarrow \text{Jac}(\Sigma)$$

is surjective and bimeromorphic. Its meromorphic inverse is described as follows: there exists a vector $\Delta \in \mathbb{C}^g$, called the vector of Riemann constants, depending only on the symplectic basis $(\mathbf{A}, \mathbf{B})$ and basepoint ζ_0 , such that for each $\xi \in \mathbb{C}^g$, exactly one of the following holds:

- (1) *The function $\zeta \mapsto \theta(\mu_{\zeta_0}(\zeta) - \xi + \Delta \mid \Pi)$ vanishes identically on Σ . In this case $[\xi]$ lies in a proper closed subset of $\text{Jac}(\Sigma)$.*

- (2) The function $\zeta \mapsto \theta(\mu_{\zeta_0}(\zeta) - \boldsymbol{\xi} + \boldsymbol{\Delta} \mid \Pi)$ has exactly g zeroes $\zeta_1, \dots, \zeta_g \in \Sigma$ (counted with multiplicity), satisfying

$$\mu_{\zeta_0}(\zeta_1 + \dots + \zeta_g) = [\boldsymbol{\xi}].$$

Remark 2.5. If $g \geq 1$, the Abel map embeds Σ into $\text{Jac}(\Sigma)$. The *theta divisor* is

$$\Theta := \{\boldsymbol{\xi} \in \mathbb{C}^g \mid \theta(\boldsymbol{\xi} \mid \Pi) = 0\}.$$

Although $\theta(\boldsymbol{\xi} \mid \Pi)$ is multi-valued on $\text{Jac}(\Sigma)$, the zero set is periodic, i.e.,

$$\Theta + (2\pi i \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g) = \Theta,$$

so Θ descends to a well-defined subset of $\text{Jac}(\Sigma)$. Riemann's theorem then asserts that for $\boldsymbol{\xi} \in \mathbb{C}^g$, either $\mu_{\zeta_0}(\Sigma) \subseteq \Theta + [\boldsymbol{\xi}] - [\boldsymbol{\Delta}]$, or else $\mu_{\zeta_0}(\Sigma) \cap (\Theta + [\boldsymbol{\xi}] - [\boldsymbol{\Delta}])$ consists of exactly g points counted with multiplicity.

2.7. Standard divisors. For Harnack curves (and more generally, M-curves), Boutillier–Cimasoni–de Tilière [BCdT23] prove stronger properties than the general statements of Section 2.5 and Section 2.6. These properties will be important for tropicalizing the theta function.

For a Harnack curve Σ° of genus g , the connected components of the real locus $\Sigma(\mathbb{R})$ are called *ovals*. There are $g + 1$ ovals in total. There is a unique *outer oval* B_0^{ov} intersecting the lines at infinity, while the remaining ovals $\boldsymbol{B} = (B_1^{\text{ov}}, \dots, B_g^{\text{ov}})$, called *compact ovals*, are in bijection with the interior lattice points of N . The compact ovals are in bijection with the bounded connected components of the amoeba; we give them the orientation induced by the amoeba. In other words, the outer oval is oriented counterclockwise and the compact ovals are oriented clockwise.

Choosing cycles $\boldsymbol{A}^{\text{ov}} = (A_1^{\text{ov}}, \dots, A_g^{\text{ov}})$ that are anti-invariant under complex conjugation, and

$$\boldsymbol{A}^{\text{ov}} \wedge \boldsymbol{B}^{\text{ov}} = I,$$

we obtain a symplectic basis $(\boldsymbol{A}^{\text{ov}}, \boldsymbol{B}^{\text{ov}})$ of $H_1(\Sigma, \mathbb{Z})$; see [KO06, Section 2.2.4]. We call such a basis *oval-adapted*.

Hereafter, we denote objects associated to the symplectic basis $(\boldsymbol{A}^{\text{ov}}, \boldsymbol{B}^{\text{ov}})$ by a superscript *ov*. We also fix a basepoint $\zeta_0 \in B_0^{\text{ov}}$ throughout.

Proposition 2.6 (Boutillier–Cimasoni–de Tilière [BCdT23]). *For a Harnack Σ° with basepoint $\zeta_0 \in B_0^{\text{ov}}$ and an oval-adapted symplectic basis $(\boldsymbol{A}^{\text{ov}}, \boldsymbol{B}^{\text{ov}})$, we have:*

- (1) If $\zeta \in B_0^{\text{ov}}$, then $\mu_{\zeta_0}(\zeta) \in \mathbb{R}^g$.
- (2) $\Pi_{jk}^{\text{ov}} \in \mathbb{R}$ for all $1 \leq j, k \leq g$.
- (3) $\boldsymbol{\Delta}^{\text{ov}} \in \pi i \mathbf{1} + \mathbb{R}^g$.

Proof. Part (1) is [BCdT23, Lemma 15(2)], part (2) is [BCdT23, Lemma 11(3)], and part (3) is [BCdT23, Lemma 19].² $\square$

²The statements in Proposition 2.6 appear different from those in [BCdT23] due to our choice of normalization $\int_{A_j} \omega_k = 2\pi i \delta_{jk}$, following [Fay73], rather than $\int_{A_j} \omega_k = \delta_{jk}$. Moreover, our convention for A - and B -cycles is also reversed compared to [BCdT23].

Following Kenyon–Okounkov [KO06], we call a degree- g effective divisor D in $\Sigma^\circ(\mathbb{R})$ a *standard divisor* if it has exactly one point in each compact oval. We denote the set of standard divisors on Σ by

$$\mathrm{Sym}_{>0}^g(\Sigma) := \prod_{j=1}^g B_j^{\mathrm{ov}}.$$

We also denote

$$\mathrm{Jac}_{>0}(\Sigma) := \pi i \mathbf{1} + (\mathbb{R}^g / \Pi^{\mathrm{ov}} \mathbb{Z}^g),$$

which is a connected component of the real part of the Jacobian $\mathrm{Jac}(\Sigma)$ translated by $\pi i \mathbf{1}$, where $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{C}^g$.

We have the following refinement of Theorem 2.4 for standard divisors.

Proposition 2.7 (Boutillier–Cimasoni–de Tilière [BCdT23, Lemma 15]). *For a Harnack curve Σ with basepoint $\zeta_0 \in B_0^{\mathrm{ov}}$ and symplectic basis $(\mathbf{A}^{\mathrm{ov}}, \mathbf{B}^{\mathrm{ov}})$ as in Section 2.4, the Abel map induces a homeomorphism*

$$\begin{aligned} \mu_{\zeta_0} : \mathrm{Sym}_{>0}^g(\Sigma) &\rightarrow \mathrm{Jac}_{>0}(\Sigma) \\ D &\mapsto \mu_{\zeta_0}(D). \end{aligned}$$

In particular, for standard divisors always correspond to Case (2) of Theorem 2.4.

2.8. Angle maps. An *angle map* is a bijection

$$\nu : Z(G) \rightarrow \Sigma \setminus \Sigma^\circ$$

between zig-zag paths and points at infinity such that for all $\alpha \in Z(G)$, we have that

$$\nu(\alpha) \in \Sigma \cap \partial V_{E^*(\alpha)}^{\mathrm{tor}}.$$

When viewed as a tuple, we write $\boldsymbol{\nu} = (\nu(\alpha))_{\alpha \in Z(G)}$. For brevity, we often write α for $\nu(\alpha)$.

2.9. The spectral transform. A *spectral datum associated to G* is a triple (Σ, D, ν) where:

- (1) Σ is a Harnack curve with Newton polygon N .³
- (2) $D \in \mathrm{Sym}_{>0}^g(\Sigma)$ is a standard divisor.
- (3) ν is an angle map.

Let $\mathbf{w}$ be a fixed reference white vertex of G . Consider the $\mathbf{w}$ -column of the adjugate matrix $\mathbf{Q}(\mathbf{z})$ of $\mathbf{K}(\mathbf{z})$. By [KO06], the points where the entries of this column simultaneously vanish, i.e.,

$$\mathbf{Q}(\mathbf{z})_{\mathbf{bw}} = 0, \quad \text{for all } \mathbf{b} \in B(G),$$

defines a standard divisor

$$D = \sum_{j=1}^g \zeta_j.$$

Following [GK13, Section 7.2] and [KO06, Section 3.1], define the *Casimirs*

$$\mathbf{C} = (C_\alpha)_{\alpha \in Z(G)}, \quad C_\alpha := [\mathrm{wt}](\alpha).$$

Since $\sum_{\alpha \in Z(G)} [\alpha] = 0$, we have $\prod_{\alpha \in Z(G)} C_\alpha = 1$. Recall that the tentacles of $\mathbb{A}_{\Sigma^\circ}$ are in bijection with the points at infinity of Σ° . For each $\alpha \in Z(G)$, there is a point at infinity

³The space of Harnack curves with Newton polygon N has four components related by $(z, w) \mapsto (\pm z, \pm w)$. The choice of Kasteleyn signs fixes the relevant component, which we use throughout.

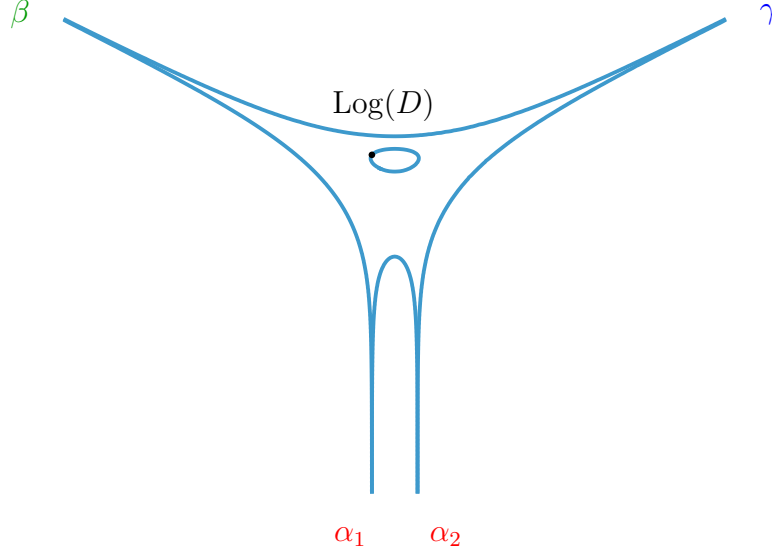


FIGURE 6. The real locus of the amoeba of the spectral curve, together with the standard divisor and angle map.

$\nu(\alpha) \in \Sigma \cap \partial V_{E^*(\alpha)}^{\text{tor}}$ determined by the property that the tentacle of $\mathbb{A}_{\Sigma^\circ}$ corresponding to a zig-zag path α with $[\alpha]$ is asymptotic to the line

$$(2.4) \quad \langle [\alpha], \mathbf{x} \rangle + \log |C_\alpha| = 0.$$

This determines an angle map

$$\nu : Z(G) \rightarrow \Sigma \cap \bigcup_{\text{sides } E^* \text{ of } N} \partial V_{E^*}^{\text{tor}}.$$

The map

$$\begin{aligned} \lambda : \{\text{positive edge weights on } G\} / \text{gauge} &\rightarrow \{\text{spectral data associated to } G\} \\ [\text{wt}] &\mapsto (\Sigma, D, \nu) \end{aligned}$$

is called the *spectral transform*.

Theorem 2.8. *The spectral transform is a bijection.*

Theorem 2.8 was originally proved for positive spectral data on the hexagonal lattice by Kenyon–Okounkov [KO06] through an indirect argument. Explicit inverses were later constructed for complex spectral data by Fock [Foc15] and George–Goncharov–Kenyon [GGK23]. The positive counterpart of Fock’s inverse map stated above for general graphs is due to Boutillier–Cimasoni–de Tilière [BCdT23].

Example 2.9. We continue with the running example and compute its spectral transform. Choose w_1 as the reference white vertex. The w_1 -column of Q is

$$Q_{\bullet, w_1}(z, w) = \begin{pmatrix} \frac{X_1 \rho_w w - 1 - X_1 X_2 X_3}{X_1 \rho_w w} \\ \frac{\rho_z z + X_1 X_2 X_3}{\rho_z \rho_w z w} \\ \frac{X_2 X_3 \rho_z z - X_2 X_3 - \rho_z \rho_w z w}{\rho_z \rho_w^2 z w^2} \\ -\frac{X_2 X_3 (X_1 \rho_w w + \rho_z z - 1)}{\rho_z \rho_w^2 z w^2} \end{pmatrix},$$

from which we get the standard divisor

$$D = \left(-\frac{X_1 X_2 X_3}{\rho_z}, \frac{1 + X_1 X_2 X_3}{X_1 \rho_w} \right).$$

We next compute the angle map. Let $\alpha_1, \alpha_2, \beta, \gamma \in Z(G)$ denote the four zig-zag paths, labelled so that

$$[\alpha_1] = [\alpha_2] = (-1, 0), \quad [\beta] = (1, 2), \quad [\gamma] = (1, -2),$$

and

$$C_{\alpha_1} = \rho_z, \quad C_{\alpha_2} = \frac{\rho_z}{X_1 X_2}, \quad C_\beta = \frac{X_2 X_3}{\rho_z \rho_w^2}, \quad C_\gamma = \frac{X_1 \rho_w^2}{X_3 \rho_z}.$$

The angle map ν is only nontrivial for the horizontal side of N . The two tentacles are along $\log |w| \rightarrow -\infty$ with

$$x + \log |C_{\alpha_1}| = 0, \quad x + \log |C_{\alpha_2}| = 0.$$

Figure 6 illustrates the spectral transform for

$$X_1 = 2, \quad X_2 = 2, \quad X_3 = 1, \quad \rho_z = 2, \quad \rho_w = 2.$$

2.10. Fock's inverse spectral transform. Via ν , we identify $\mathbb{Z}^{Z(G)}$ with $\mathbb{Z}^{\Sigma \setminus \Sigma^\circ}$, i.e., the group of divisors supported on $\Sigma \setminus \Sigma^\circ \subset B_0^{\text{ov}}$. The *discrete Abel map*

$$\mathbf{d} : B(\tilde{G}) \sqcup W(\tilde{G}) \sqcup F(\tilde{G}) \rightarrow \mathbb{Z}^{Z(G)}$$

assigns to each vertex and face of $\tilde{G}$ a divisor supported on $Z(G)$ according to the following rules:

- (1) Normalize by setting $\mathbf{d}(\tilde{w}) = 0$ for a reference white vertex $\tilde{w}$.
- (2) For a face $\tilde{f}$ incident to a black vertex $\tilde{b}$, set

$$\mathbf{d}(\tilde{b}) - \mathbf{d}(\tilde{f}) = \alpha,$$

where α is the zig-zag path in G whose lift to $\tilde{G}$ separates $\tilde{f}$ and $\tilde{b}$.

- (3) For a face $\tilde{f}$ incident to a white vertex $\tilde{w}$, set

$$\mathbf{d}(\tilde{w}) - \mathbf{d}(\tilde{f}) = -\alpha,$$

where α is the zig-zag path in G whose lift to $\tilde{G}$ separates $\tilde{f}$ and $\tilde{w}$.

Recall that we fix a basepoint $\zeta_0 \in B_0^{\text{ov}}$. Let $\tilde{\Sigma}$ denote the universal cover of Σ . We fix a lift $\tilde{\zeta}_0$ of ζ_0 and for each zig-zag path $\alpha \in Z(G)$, we fix a lift $\tilde{\alpha} \in \tilde{\Sigma}$. Since $\tilde{\Sigma}$ is simply connected, any two paths in $\tilde{\Sigma}$ joining $\tilde{\zeta}$ to $\tilde{\zeta}'$ are homotopic relative to their endpoints. For a 1-form ω , we use the convention that

$$\int_{\tilde{\zeta}}^{\tilde{\zeta}'} \omega$$

denotes the integral along any path in $\tilde{\Sigma}$ joining the chosen lift of ζ to the chosen lift of ζ' . For a holomorphic 1-form, this integral is independent of the chosen path. For the meromorphic differentials of the third kind below, we also assume that the path avoids the poles. Although the integrals may depend on the path, we only consider their exponentials which are again path independent. Unless stated otherwise, all integrals in this section are understood in this sense.

A *wedge* of G is a 1-chain

$$\mathfrak{w} = \mathbf{e}_+ - \mathbf{e}_-,$$

where $\mathbf{e}_- = \mathbf{b}\mathbf{w}_-$, and $\mathbf{e}_+ = \mathbf{b}\mathbf{w}_+$ are two edges incident to a common black vertex $\mathbf{b}$. Let $\alpha, \beta_-, \beta_+ \in Z(G)$ be the zig-zag paths determined by

$$\mathbf{e}_- \in \alpha \cap \beta_-, \quad \mathbf{e}_+ \in \alpha \cap \beta_+.$$

Let $\mathbf{f}_-$, respectively $\mathbf{f}_+$, be the face separated from $\mathbf{b}$ by β_- , respectively β_+ ; see Figure 3.

Let (Σ, D, ν) be a spectral datum associated to G . We assign to a $\mathfrak{w}$ the *wedge weight*

$$(2.5) \quad \text{wt}(\mathfrak{w}) := -\frac{\theta(\mu_{\zeta_0}^{\text{ov}}(\mathbf{d}(\mathbf{f}_-)) - \mu_{\zeta_0}^{\text{ov}}(D) + \Delta^{\text{ov}} \mid \Pi^{\text{ov}})}{\theta(\mu_{\zeta_0}^{\text{ov}}(\mathbf{d}(\mathbf{f}_+)) - \mu_{\zeta_0}^{\text{ov}}(D) + \Delta^{\text{ov}} \mid \Pi^{\text{ov}})} \exp \left(\int_{\zeta_0}^{\alpha} \omega_{\beta_+ - \beta_-}^{\text{ov}} \right).$$

Here, $\omega_{\beta_+ - \beta_-}^{\text{ov}}$ is the A -normalized differential of the third kind with residue divisor $\beta_+ - \beta_-$, and the Abel maps and $\int_{\zeta_0}^{\alpha} \omega_{\beta_+ - \beta_-}^{\text{ov}}$ use our convention for integrals above.

Any cycle γ in G can be decomposed as a sum of wedges,

$$\gamma = \mathfrak{w}_1 + \cdots + \mathfrak{w}_r.$$

We define its cycle weight by

$$[\text{wt}](\gamma) := [\kappa](\gamma) \prod_{j=1}^r \text{wt}(\mathfrak{w}_j),$$

where $[\kappa]$ is the gauge equivalence class of the Kasteleyn signs. In particular, the face weights are given by

$$X_{\mathbf{f}} := (-1)^{\frac{|\partial \mathbf{f}|}{2} + 1} \prod_{j=1}^r \text{wt}(\mathfrak{w}_j).$$

The corresponding gauge-equivalence class of edge weights on G are called *Fock weights*.

Theorem 2.10 (Fock [Foc15]). *The assignment of Fock weights to spectral data inverts the spectral transform.*

3. TROPICAL SPECTRAL CURVES

3.1. Puiseux series. Throughout, we use the max convention for tropicalization. The *field of formal Puiseux series* is

$$\mathbb{C}\{\{t^{-1}\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{-1/n})).$$

An element $c(t) \in \mathbb{C}\{\{t^{-1}\}\}$ of this field has the form

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots,$$

where $c_i \neq 0$ and $\cdots < a_2 < a_1 \in \mathbb{Q}$ have a common denominator. The coefficient c_1 is called the *leading coefficient* and is denoted by $\text{lc}(c(t))$. The field $\mathbb{C}\{\{t^{-1}\}\}$ carries the valuation

$$\begin{aligned} \text{val} : \mathbb{C}\{\{t^{-1}\}\} &\rightarrow \mathbb{Q} \cup \{-\infty\}, \\ c(t) &\mapsto a_1, \end{aligned}$$

with $\text{val}(0) := -\infty$.

Let $\mathbb{K} \subset \mathbb{C}\{\{t^{-1}\}\}$ be the subfield of *convergent Puiseux series*, i.e., those Puiseux series that have a positive radius of convergence as $t \rightarrow \infty$. The semifield $\mathbb{K}_{>0}$ of *positive convergent Puiseux series* consists of those $c(t) \in \mathbb{K}$ that are convergent, have real coefficients, and whose leading coefficient is positive.

Throughout this paper, we will only consider convergent Puiseux series. Therefore, we will simply refer to them as Puiseux series and omit the qualifier “convergent”.

3.2. Tropicalization. Let $[\text{wt}](t)$ be $\mathbb{K}_{>0}$ -valued edge weights modulo gauge equivalence, and let

$$P(\mathbf{z}; t) = \sum_{\mathbf{p} \in N_{\mathbb{Z}}} c_{\mathbf{p}}(t) \mathbf{z}^{\mathbf{p}}$$

be the corresponding family of characteristic polynomials. Set $c_{\mathbf{p}}^{\dagger} := \text{val}(c_{\mathbf{p}}(t))$. The *tropical characteristic polynomial* is

$$(3.1) \quad P^{\dagger}(\mathbf{x}) := \max_{\mathbf{p} \in N_{\mathbb{Z}}} (\langle \mathbf{p}, \mathbf{x} \rangle + c_{\mathbf{p}}^{\dagger}),$$

and the *embedded tropical spectral curve* is

$$(3.2) \quad \Sigma_{\text{emb}}^{\dagger} := \{\mathbf{x} \in \mathbb{R}^2 \mid P^{\dagger}(\mathbf{x}) \text{ achieves its maximum at least twice}\}.$$

Let $\Sigma(t)$ be the corresponding family of spectral curves. By the fundamental theorem of tropical geometry [Mik04, Theorem 4.6],

$$(3.3) \quad \Sigma_{\text{emb}}^{\dagger} = \lim_{t \rightarrow \infty} \frac{1}{\log t} \text{Log}(\Sigma(t)),$$

where the limit is taken in the Hausdorff metric on subsets of $\mathbb{R}^2$.

The function

$$c^{\dagger} : N_{\mathbb{Z}} \rightarrow \mathbb{R}$$

induces a regular subdivision of N in the sense of [GKZ08]. Namely, consider the upper hull

$$(3.4) \quad \text{conv}\{(\mathbf{p}; z) \in N_{\mathbb{Z}} \times \mathbb{R} \mid z \leq c_{\mathbf{p}}^{\dagger}\}.$$

The cells of the subdivision are the projections to N of the upper faces of (3.4). The subdivision of $\mathbb{R}^2$ into the domains of linearity of $P^{\dagger}$ and the regular subdivision are dual polyhedral complexes; see [MS15, Proposition 3.1.6] and Figure 8 for an example.

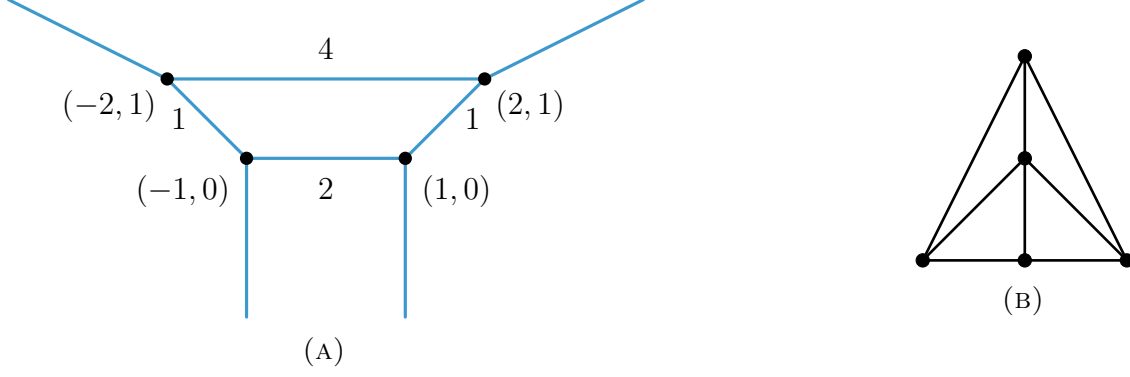


FIGURE 7. The embedded tropical spectral curve associated to $P^\dagger(x, y) = \max\{0, 1-y, 1-2y, -x-2y, x-2y\}$ (A) and the corresponding dual subdivision of its Newton polygon (B). The bounded edges of the tropical curve are labeled by their lattice lengths.

Example 3.1. We return to the running example of Example 2.2 and choose

$$X_1(t) = t, \quad X_2(t) = t, \quad X_3(t) = 1, \quad \rho_z(t) = t, \quad \rho_w(t) = 1.$$

The limit (3.3) is illustrated in Figure 1. The tropical characteristic polynomial is

$$P^\dagger(x, y) = \max\{0, 1-y, 1-2y, -x-2y, x-2y\}.$$

Its tropical curve is shown in Figure 7.

3.3. Abstract tropical curves. An *abstract tropical curve* is a triple $(\Sigma^\dagger, \ell, \mathbf{g})$, where:

- $\Sigma^\dagger$ is a finite connected graph.
- $\ell : E(\Sigma^\dagger) \rightarrow \mathbb{R}_{>0}$ is a *length* function.
- $\mathbf{g} = (g_v)_{v \in V(\Sigma^\dagger)}$, $g_v \in \mathbb{Z}_{\geq 0}$, is a genus assigned to each vertex.

Set

$$g^{\text{comp}} := \sum_{v \in V(\Sigma^\dagger)} g_v, \quad g^\dagger := \text{rank } H_1(\Sigma^\dagger, \mathbb{Z}), \quad g := g^{\text{comp}} + g^\dagger.$$

We call g the *genus* of the abstract tropical curve.

For a line segment $I = [a, b] \subset \mathbb{R}^2$ parallel to a primitive vector $u \in \mathbb{Z}^2$, define its *lattice length* $|I|_{\mathbb{Z}}$ by $b - a = |I|_{\mathbb{Z}} u$. We now associate an abstract tropical curve to (3.2). For each bounded edge $\bar{e}$ of $\Sigma_{\text{emb}}^\dagger$, let $\bar{e}^*$ be its dual edge and set

$$m_{\bar{e}} := |\bar{e}^*|_{\mathbb{Z}}.$$

The underlying graph $\Sigma^\dagger$ is obtained from $\Sigma_{\text{emb}}^\dagger$ by contracting all unbounded edges and replacing each bounded edge $\bar{e}$ by $m_{\bar{e}}$ distinct edges $e_1, \dots, e_{m_{\bar{e}}}$ with the same endpoints. Thus, there is a natural map

$$\pi : \Sigma^\dagger \rightarrow \Sigma_{\text{emb}}^\dagger$$

which is the identity on vertices and maps each edge $e_i \in \pi^{-1}(\bar{e})$ linearly onto $\bar{e}$. In particular, $V(\Sigma^\dagger) = V(\Sigma_{\text{emb}}^\dagger)$. For each edge e_i lying over $\bar{e}$, define $\ell(e_i) := |\bar{e}|_{\mathbb{Z}}$. For $v \in V(\Sigma^\dagger)$, let v^* be the dual polygon and set $g_v := |(v^*)^\circ \cap \mathbb{Z}^2|$. This defines the abstract tropical curve $(\Sigma^\dagger, \ell, \mathbf{g})$.

In our setting, $\Sigma^\dagger$ comes with a natural planar embedding. For each bounded embedded edge $\bar{e}$, draw the edges in $\pi^{-1}(\bar{e})$ as parallel edges in a neighborhood of $\bar{e}$. Their ordering will

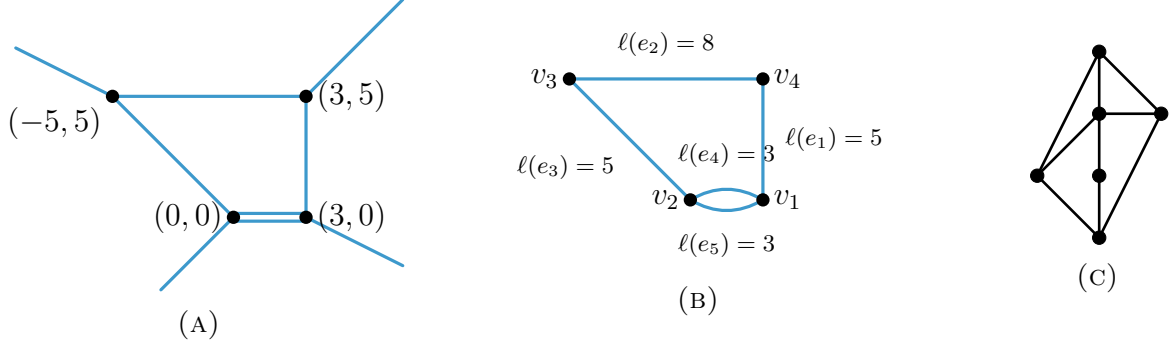


FIGURE 8. An embedded tropical curve with an edge of multiplicity 2 (A), the corresponding abstract tropical curve obtained by replacing the multiplicity-two edge with a bigon and contracting unbounded edges (B), and the dual subdivision of the Newton polygon (C).

be identified in Section 4.1 with the cyclic order of the corresponding nodes in the special fiber of the family $(\Sigma(t))_{t \gg 1}$.

With respect to this planar embedding, $\Sigma^\dagger$ has $g^\dagger + 1$ faces. Let $f_0 \in F(\Sigma^\dagger)$ denote the outer face and set

$$F^\circ(\Sigma^\dagger) := F(\Sigma^\dagger) \setminus \{f_0\}.$$

We orient the boundary of f_0 counterclockwise and the boundaries of all other faces clockwise, and define $B_f^{\text{ov}, \dagger} := \partial f$. The cycles $\mathbf{B}^{\text{ov}, \dagger} = (B_f^{\text{ov}, \dagger})_{f \in F^\circ(\Sigma^\dagger)}$ form a basis of $H_1(\Sigma^\dagger, \mathbb{Z})$, and $\sum_{f \in F(\Sigma^\dagger)} B_f^{\text{ov}, \dagger} = 0$.

Example 3.2. Consider the embedded tropical curve shown in Figure 8(A). The bounded edge joining $(0, 0)$ to $(3, 0)$ has multiplicity two. Passing to the corresponding abstract tropical curve, we replace this multiplicity-two edge by a bigon. The vertices all have genus 0 and the lengths of the bounded edges are indicated in Figure 8(B).

3.4. Tropical differentials. Let $(\Sigma^\dagger, \ell, \mathbf{g})$ be a abstract tropical curve of genus g . Let $H(\Sigma^\dagger)$ denote the set of half-edges of $\Sigma^\dagger$. For each $h \in H(\Sigma^\dagger)$, let $v(h)$ be its incident vertex, let $e(h)$ be its underlying unoriented edge, and let $\bar{h}$ be the opposite half-edge. Thus, $e(h) = e(\bar{h})$. For each vertex $v \in V(\Sigma^\dagger)$, set

$$H_v := \{h \in H(\Sigma^\dagger) \mid v(h) = v\}.$$

A *meromorphic tropical 1-form* on the abstract tropical curve $(\Sigma^\dagger, \ell, \mathbf{g})$ is an antisymmetric function

$$\omega : H(\Sigma^\dagger) \rightarrow \mathbb{Z}, \quad \omega(\bar{h}) = -\omega(h).$$

The *residue* of ω at a vertex $v \in V(\Sigma^\dagger)$ is

$$\text{res}_v(\omega) := \sum_{h \in H_v} \omega(h).$$

A vertex v is called a *pole* of ω if $\text{res}_v(\omega) \neq 0$. We say that ω is *holomorphic* if $\text{res}_v(\omega) = 0$ for every $v \in V(\Sigma^\dagger)$.

Let I be an oriented edge segment contained in the edge $e(h)$ and oriented in the direction of the half-edge h . We define the *tropical integral*

$$\int_I \omega := \omega(h) \ell(I),$$

where $\ell(I)$ denotes the length of I in the metric induced by ℓ . The tropical integral extends to 1-chains by linearity.

3.5. Tropical Abel map and Jacobian. Let $(\Sigma^\dagger, \ell, \mathbf{g})$ be a abstract tropical curve of genus g . Fix a basis $\mathbf{C}^\dagger = (C_1^\dagger, \dots, C_{g^\dagger}^\dagger)$ of $H_1(\Sigma^\dagger, \mathbb{Z})$. For any $1 \leq j \leq g^\dagger$, let $\omega_j^\dagger$ to be the holomorphic tropical 1-form such that, for every $h \in H(\Sigma^\dagger)$, $\omega_j^\dagger(h)$ is the coefficient of the edge $e(h)$, oriented in the direction of h , in the cycle $C_j^\dagger$. In other words, $\omega_j^\dagger$ is the unit flow around $C_j^\dagger$.

The *tropical period matrix* is the positive-definite $g^\dagger \times g^\dagger$ matrix

$$\Pi^\dagger := \int_{\mathbf{C}^\dagger} \omega^\dagger.$$

The *tropical Jacobian* is the real $g^\dagger$ -dimensional torus

$$\text{Jac}(\Sigma^\dagger) := \mathbb{R}^{g^\dagger} / \Pi^\dagger \mathbb{Z}^{g^\dagger}.$$

For $\boldsymbol{\xi}^\dagger \in \mathbb{R}^{g^\dagger}$, we denote its image in $\text{Jac}(\Sigma^\dagger)$ by $[\boldsymbol{\xi}^\dagger]$.

The *tropical Abel map*

$$\mu_{\zeta_0^\dagger}^\dagger : \Sigma^\dagger \rightarrow \text{Jac}(\Sigma^\dagger)$$

with basepoint $\zeta_0^\dagger \in \Sigma^\dagger$ is defined by

$$\mu_{\zeta_0^\dagger}^\dagger(\zeta^\dagger) := \int_{\zeta_0^\dagger}^{\zeta^\dagger} \omega^\dagger \mod \Pi^\dagger \mathbb{Z}^{g^\dagger}.$$

The tropical Abel map extends linearly to divisors.

A natural choice of basis is $\mathbf{C}^\dagger = \mathbf{B}^{\text{ov}, \dagger}$, in which case we denote the corresponding objects by an ov in the superscript. However, it will be more convenient to work with a different basis below.

Example 3.3. Consider the abstract tropical curve in Figure 8(B). It has two bounded faces. Let $\mathbf{B}^{\text{ov}, \dagger} = (B_1^{\text{ov}, \dagger}, B_2^{\text{ov}, \dagger})$, where $B_1^{\text{ov}, \dagger}$ is the clockwise boundary of the bigon and $B_2^{\text{ov}, \dagger}$ is the clockwise boundary of the other bounded face. Then,

$$\int_{B_1^{\text{ov}, \dagger}} \omega_1^{\text{ov}, \dagger} = 3 + 3 = 6, \quad \int_{B_2^{\text{ov}, \dagger}} \omega_2^{\text{ov}, \dagger} = 3 + 5 + 8 + 5 = 21,$$

whereas

$$\int_{B_1^{\text{ov}, \dagger}} \omega_2^{\text{ov}, \dagger} = \int_{B_2^{\text{ov}, \dagger}} \omega_1^{\text{ov}, \dagger} = -3.$$

Therefore, the tropical period matrix in the oval-adapted basis is

$$\Pi^{\text{ov}, \dagger} = \int_{\mathbf{B}^{\text{ov}, \dagger}} \omega^{\text{ov}, \dagger} = \begin{pmatrix} 6 & -3 \\ -3 & 21 \end{pmatrix}.$$

4. DEGENERATION OF THE SPECTRAL CURVE

Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth genus- g curves with n marked points, i.e., pairs $(\Sigma, \boldsymbol{\nu} = (\nu_1, \dots, \nu_n))$ where Σ is a smooth curve of genus g and $\nu_1, \dots, \nu_n$ are distinct points of Σ . For brevity, when g and n are understood from context, we refer to $(\Sigma, \boldsymbol{\nu})$ as a *pointed curve*.

Its Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ is obtained by allowing stable curves. A *stable pointed curve* is a connected compact curve Σ whose only singularities are nodes, together with n distinct marked smooth points $\boldsymbol{\nu}$, satisfying a stability condition on each irreducible component. More precisely, let Σ_v be a component of the normalization, let g_v be its genus, and let n_v be the number of special points on Σ_v , consisting of the preimages of the nodes and the marked points. The stability condition is

$$2g_v - 2 + n_v > 0$$

for every component Σ_v . In other words, every genus-0 component has at least three special points, and every genus-1 component has at least one special point.

The relevance of the Deligne–Mumford compactification for us is that the family of pointed spectral curves $(\Sigma(t), \boldsymbol{\nu}(t))$ has a limit in $\overline{\mathcal{M}}_{g,n}$. We will show that this limit is a stable pointed curve $(\Sigma(\infty), \boldsymbol{\nu}(\infty))$, whose irreducible components are naturally indexed by the vertices of the tropical spectral curve and whose nodes correspond to its bounded edges. We then describe $\Sigma(t)$ near this boundary point of $\overline{\mathcal{M}}_{g,n}$ using plumbing coordinates. This description will allow us to study the asymptotics of the classical objects on $\Sigma(t)$ that enter Fock’s formula using results of [HN20].

4.1. Toric degeneration. Let $[\text{wt}](t)$ be a $\mathbb{K}_{>0}$ -valued edge weight modulo gauge equivalence satisfying a genericity condition that we will specify later, and let

$$(\Sigma(t))_{t \gg 1} \subset V_N^{\text{tor}}$$

be the corresponding family of spectral curves with embedded tropical spectral curve $\Sigma_{\text{emb}}^\dagger$. Then $\Sigma_{\text{emb}}^\dagger$ determines a degeneration of V_N^{tor} inside which the family $\Sigma(t)$ itself degenerates to a nodal curve $\Sigma(\infty)$. In this section, we describe this degeneration; see for example [MS15, Section 6.6] for details.

By replacing t with $t^{\frac{1}{M}}$ for some large $M \in \mathbb{Z}$, we can assume that $c_{\mathbf{p}}^\dagger \in \mathbb{Z}$ for all $\mathbf{p} \in N_{\mathbb{Z}}^\circ$. The tropical curve $\Sigma_{\text{emb}}^\dagger$ determines a *toric degeneration*

$$p : \mathcal{V}_{\Upsilon}^{\text{tor}} \rightarrow \mathbb{C},$$

where

$$\mathcal{V}_{\Upsilon}^{\text{tor}} := \overline{\left\{ ([t^{\frac{1}{M}} \mathbf{z}^{\mathbf{p}}], t^{-1})_{\mathbf{p} \in N_{\mathbb{Z}}} \mid \mathbf{z} \in (\mathbb{C}^\times)^2 \right\}} \subset \mathbb{CP}^{|N_{\mathbb{Z}}|-1} \times \mathbb{C}$$

and p is projection to the second factor $\mathbb{C}$. Let

$$\mathcal{V}_{\Upsilon}^{\text{tor}}(t) := p^{-1}(t^{-1})$$

denote the fibers. When $t \neq \infty$, $\mathcal{V}_{\Upsilon}^{\text{tor}}(t)$ is isomorphic to V_N^{tor} . When $t = \infty$ (i.e., when $t^{-1} = 0$),

$$\mathcal{V}_{\Upsilon}^{\text{tor}}(\infty) = \bigcup_{v \in V(\Sigma^\dagger)} V_{v^*}^{\text{tor}},$$

where the toric surfaces $V_{v^*}^{\text{tor}}$ intersect along their lines at infinity according to the dual regular subdivision.

Inside $\mathbb{CP}^{|N_{\mathbb{Z}}|-1}$, define the hypersurface

$$H(t) := \left\{ [\mathbf{z}_{\mathbf{p}}]_{\mathbf{p} \in N_{\mathbb{Z}}} \left| \sum_{\mathbf{p} \in N_{\mathbb{Z}}} t^{-c_{\mathbf{p}}^{\dagger}} c_{\mathbf{p}}(t) \mathbf{z}_{\mathbf{p}} = 0 \right. \right\}.$$

For $t \neq \infty$, the spectral curve is the intersection

$$\Sigma(t) = \mathcal{V}_{\Upsilon}^{\text{tor}}(t) \cap H(t).$$

When $t = \infty$, intersecting with the hypersurface

$$H(\infty) := \left\{ [\mathbf{z}_{\mathbf{p}}]_{\mathbf{p} \in N_{\mathbb{Z}}} \left| \sum_{\mathbf{p} \in N_{\mathbb{Z}}} \text{lc}(c_{\mathbf{p}}(t)) \mathbf{z}_{\mathbf{p}} = 0 \right. \right\}$$

gives the special fiber

$$\Sigma(\infty) := \mathcal{V}_{\Upsilon}^{\text{tor}}(\infty) \cap H(\infty).$$

The curve

$$\Sigma(\infty) = \bigcup_{v \in V(\Sigma^{\dagger})} \Sigma_v, \quad \Sigma_v := V_{v^*}^{\text{tor}} \cap \Sigma(\infty).$$

Suppose $\mathbf{x} \in \mathbb{R}^2$ are the coordinates of v . Then, Σ_v is a Harnack curve defined inside $V_{v^*}^{\text{tor}}$, cut out by the *initial form*

$$\text{init}_v(P)(\mathbf{z}) := \lim_{t \rightarrow \infty} t^{-P^{\dagger}(\mathbf{x})} P(t^{\mathbf{x}} \mathbf{z}; t).$$

Assumption 4.1. We assume that $[\text{wt}](t)$ is generic in the sense that each Σ_v is smooth and transverse to the toric boundary. This is a Zariski-open condition on the leading coefficients of $[\text{wt}](t)$.

Let $\bar{e}$ be a bounded edge of $\Sigma_{\text{emb}}^{\dagger}$, with endpoints u and v , and let $\bar{e}^*$ be its dual edge. Recall that

$$\pi^{-1}(\bar{e}) = \{e_1, \dots, e_{m_{\bar{e}}}\}.$$

Without loss of generality, after a monomial change of coordinates, assume that

$$\bar{e} = [(-|\bar{e}|_{\mathbb{Z}}, 0), (0, 0)], \quad \bar{e}^* = [(0, 0), (0, m_{\bar{e}})].$$

The edge $\bar{e}$ defines an affine chart of $\mathcal{V}_{\Upsilon}^{\text{tor}}$ of the form

$$\mathcal{U}_{\bar{e}} = \{(z_u, z_v, w, t^{-1}) \in \mathbb{C}^2 \times \mathbb{C}^{\times} \times \mathbb{C} \mid z_u z_v - t^{-|\bar{e}|_{\mathbb{Z}}} = 0\},$$

where

$$z_u = z, \quad z_v = \frac{t^{-|\bar{e}|_{\mathbb{Z}}}}{z},$$

and

$$(4.1) \quad \mathcal{U}_{\bar{e}} \cap (V_{u^*}^{\text{tor}} \cap V_{v^*}^{\text{tor}}) = \{(z_u, z_v, w, t^{-1}) \in \mathcal{U}_{\bar{e}} \mid z_u = z_v = t^{-1} = 0\}.$$

In these coordinates, the equation of $\Sigma(t)$ is of the form

$$F(z_u, z_v, w, t^{-1}) = f_{\bar{e}}(w) + R(z_u, z_v, w, t^{-1}),$$

where

$$f_{\bar{e}}(w) = \sum_{k=0}^{m_{\bar{e}}} \text{lc}(c_{(0,k)}(t)) w^k$$

is a polynomial in w of degree $m_{\bar{e}}$, and $R(z_u, z_v, w, t^{-1})$ consists of terms that vanish on (4.1). By Assumption 4.1, $f_{\bar{e}}(w)$ has $m_{\bar{e}}$ distinct roots $w_1, \dots, w_{m_{\bar{e}}} \in \mathbb{C}^\times$. Let $n_i := (0, 0, w_i, 0)$. Then,

$$\Sigma_u \cap \Sigma_v = \{n_i \mid i = 1, \dots, m_{\bar{e}}\}.$$

Since

$$\frac{\partial F}{\partial w}(0, 0, w_i, 0) = f'_{\bar{e}}(w_i) \neq 0,$$

the implicit function theorem allows us, in a neighborhood of each n_i , to eliminate w , writing w as a holomorphic function of z_u, z_v, t^{-1} . Intersecting with $\mathcal{U}_{\bar{e}}$ leaves

$$z_u z_v = t^{-|\bar{e}|_Z},$$

which is the standard local model of a nodal degeneration. Thus, the points $n_1, \dots, n_{m_{\bar{e}}}$ at which Σ_u and Σ_v meet are nodes.

We identify the nodes $n_1, \dots, n_{m_{\bar{e}}}$ with the edges $e_1, \dots, e_{m_{\bar{e}}}$ lying over $\bar{e}$ so that the cyclic order of the nodes along the real locus $\Sigma_u(\mathbb{R})$ agrees with that of the edges around u .

We now regard this as a degeneration of pointed curves. For $t \gg 1$, let $\boldsymbol{\nu}(t) = (\nu(t)(\alpha))_{\alpha \in Z(G)}$ be the angle map associated with $[\text{wt}](t)$. Thus, $(\Sigma(t), \boldsymbol{\nu}(t))$ is a pointed curve, whose marked points are precisely the points at infinity of $\Sigma(t)$. Under the toric degeneration, we obtain a pointed nodal curve $(\Sigma(\infty), \boldsymbol{\nu}(\infty))$, which we claim is stable. For a component Σ_v , the number of special points $n_v = |\partial v^* \cap \mathbb{Z}^2|$ and the genus $g_v = |(v^*)^\circ \cap \mathbb{Z}^2|$. By Pick's theorem,

$$2g_v - 2 + n_v = 2 \text{Area}(v^*) > 0.$$

Thus, every component satisfies the stability condition and $(\Sigma(\infty), \boldsymbol{\nu}(\infty))$ is a stable pointed curve.

4.2. Plumbing. Let $(\Sigma(\infty), \boldsymbol{\nu}(\infty))$ be a stable pointed curve and let $(\Sigma^\dagger, \ell, \mathbf{g})$ be the abstract tropical curve. Plumbing provides a local chart around $(\Sigma(\infty), \boldsymbol{\nu}(\infty)) \in \overline{\mathcal{M}}_{g,n}$. There are two types of local parameters.

The first type vary the components of the normalization while preserving the nodes. For each $v \in V(\Sigma^\dagger)$, let $(\Sigma_v, \boldsymbol{\nu}_v) \in \mathcal{M}_{g_v, n_v}$ denote the corresponding component, where $\boldsymbol{\nu}_v$ is the tuple of special points on Σ_v which consist of the preimages of the nodes together with the original marked points lying on Σ_v . Let $U_v \subset \mathcal{M}_{g_v, n_v}$ be a sufficiently small neighborhood of $(\Sigma_v, \boldsymbol{\nu}_v)$, and set $U := \prod_{v \in V(\Sigma^\dagger)} U_v$. We write $\mathbf{u} = (u_v)_{v \in V(\Sigma^\dagger)} \in U$ for these parameters chosen so that $u_v = 0$ corresponds to $(\Sigma_v, \boldsymbol{\nu}_v)$. For $\mathbf{u} \in U$, denote the corresponding component pointed curves by $(\Sigma_v(\mathbf{u}), \boldsymbol{\nu}_v(\mathbf{u}))$. For each half-edge $h \in H_v$, we denote by $n_h(\mathbf{u})$ the marked point corresponding to the preimage of the node indexed by h .

The second type of parameters smooth the nodes. For each edge $e \in E(\Sigma^\dagger)$, let $q_e \in \mathbb{C}$ be the corresponding *plumbing parameter*, and write $\mathbf{q} = (q_e)_{e \in E(\Sigma^\dagger)}$.

We now recall the plumbing construction explicitly. After shrinking U , for each half-edge $h \in H(\Sigma^\dagger)$ we may choose a local holomorphic coordinate $z_h(\mathbf{u})$ centered at $n_h(\mathbf{u})$ which varies holomorphically with $\mathbf{u}$; see [HK14, Section 8.1]. Choose $\rho > 0$ sufficiently small so that, for every $\mathbf{u} \in U$ and every $v \in V(\Sigma^\dagger)$, the coordinate disks

$$\{|z_h(\mathbf{u})| < \rho\}, \quad h \in H_v,$$

are pairwise disjoint and contain no other points of $\boldsymbol{\nu}_v(\mathbf{u})$.

Suppose first that $q_e \neq 0$ for every $e \in E(\Sigma^\dagger)$. For each half-edge $h \in H(\Sigma^\dagger)$, remove the closed disk

$$(4.2) \quad D_h(\mathbf{u}, \mathbf{q}) := \left\{ |z_h(\mathbf{u})| \leq \frac{|q_{e(h)}|}{\rho} \right\} \subset \Sigma_{v(h)}(\mathbf{u})$$

and consider the *collar*

$$K_h(\mathbf{u}, \mathbf{q}) := \left\{ \frac{|q_{e(h)}|}{\rho} < |z_h(\mathbf{u})| < \rho \right\}.$$

For each edge $e = \{h, \bar{h}\}$, glue the collars $K_h(\mathbf{u}, \mathbf{q})$ and $K_{\bar{h}}(\mathbf{u}, \mathbf{q})$ by

$$(4.3) \quad z_h(\mathbf{u}) \mapsto z_{\bar{h}}(\mathbf{u}) = \frac{q_e}{z_h(\mathbf{u})}.$$

We denote the resulting pointed curve by $(\Sigma(\mathbf{u}, \mathbf{q}), \boldsymbol{\nu}(\mathbf{u}, \mathbf{q}))$, where

$$\Sigma(\mathbf{u}, \mathbf{q}) := \left(\bigsqcup_{v \in V(\Sigma^\dagger)} \left(\Sigma_v(\mathbf{u}) \setminus \bigsqcup_{h \in H_v} D_h(\mathbf{u}, \mathbf{q}) \right) \right) / \sim,$$

where $\sim$ is the equivalence relation generated by the gluing maps (4.3), and the points in $\boldsymbol{\nu}(\mathbf{u}, \mathbf{q})$ are the original marked points among the tuples $\boldsymbol{\nu}_v(\mathbf{u})$; the points $n_h(\mathbf{u})$ are no longer marked after the corresponding node is smoothed.

If $q_e = 0$, the node corresponding to $e = \{h, \bar{h}\}$ is left unsmoothed, so that $n_h(\mathbf{u}) \sim n_{\bar{h}}(\mathbf{u})$. In particular,

$$\Sigma(\mathbf{u}, \mathbf{0}) = \left(\bigsqcup_{v \in V(\Sigma^\dagger)} \Sigma_v(\mathbf{u}) \right) / (n_h(\mathbf{u}) \sim n_{\bar{h}}(\mathbf{u}) \text{ for every } e = \{h, \bar{h}\}),$$

and

$$(\Sigma(\mathbf{0}, \mathbf{0}), \boldsymbol{\nu}(\mathbf{0}, \mathbf{0})) = (\Sigma(\infty), \boldsymbol{\nu}(\infty)).$$

For $\mathbf{u}$ and $\mathbf{q}$ sufficiently close to $(\mathbf{0}, \mathbf{0})$, the parameters $(\mathbf{u}, \mathbf{q})$ give a local chart for a neighborhood of $(\Sigma(\infty), \boldsymbol{\nu}(\infty))$ in $\overline{\mathcal{M}}_{g,n}$; see [HK14, Sections 8–10]. Therefore, for $t^{-1} \in [0, \epsilon)$ for some $\epsilon > 0$, there is an isomorphism of pointed curves

$$(\Sigma(t), \boldsymbol{\nu}(t)) \cong (\Sigma(\mathbf{u}(t), \mathbf{q}(t)), \boldsymbol{\nu}(\mathbf{u}(t), \mathbf{q}(t)))$$

for some $(\mathbf{u}(t), \mathbf{q}(t)) \rightarrow (\mathbf{0}, \mathbf{0})$ as $t \rightarrow \infty$. By [ACP15, Definition 1.1.1 and the paragraph following it], the valuation of a smoothing parameter is independent of the local defining equation. Hence,

$$\text{val}(q_e(t)) = -\ell(e) \quad \text{for every } e \in E(\Sigma^\dagger).$$

Hereafter, since we restrict the plumbing construction to the one-parameter family $(\mathbf{u}(t), \mathbf{q}(t))$, we suppress the parameters $(\mathbf{u}(t), \mathbf{q}(t))$ from the notation and indicate the resulting dependence simply by t .

4.3. Cores and degeneration-adapted paths. Following [HK14, Proof of Theorem 3.6], for each vertex $v \in V(\Sigma^\dagger)$, define

$$\Sigma_v^{\text{core}}(t) := \Sigma_v(t) \setminus \bigsqcup_{h \in H_v} \{|z_h(t)| < \rho\}, \quad t^{-1} \in [0, \varepsilon).$$

Let $\sigma(t)$ (resp. σ_v) denote complex conjugation on $\Sigma(t)$ (resp. Σ_v). Since the nodes lie on the real locus, we may choose the local coordinate $z_h(t)$ to be compatible with complex conjugation, i.e., so that $z_h(t) \circ \sigma(t) = \overline{z_h(t)}$. Complex conjugation makes

$$\bigsqcup_{t^{-1} \in [0, \varepsilon)} \Sigma_v^{\text{core}}(t) \rightarrow [0, \varepsilon)$$

a $\mathbb{Z}/2\mathbb{Z}$ -equivariant map, where the action on the base is trivial.

After shrinking ε if necessary, the equivariant form of Ehresmann's fibration theorem for manifolds with corners [GIMM23, Lemma 2.34] gives smooth diffeomorphisms

$$(4.4) \quad \Phi = (\Phi_v)_{v \in V(\Sigma^\dagger)}, \quad \Phi_v : \Sigma_v^{\text{core}}(\infty) \times [0, \varepsilon) \rightarrow \bigsqcup_{t^{-1} \in [0, \varepsilon)} \Sigma_v^{\text{core}}(t), \quad \Phi_v(\cdot; 0) = \text{id},$$

such that:

- For every $h \in H_v$, Φ_v is compatible with the boundary trivializations provided by $z_h(t)$, i.e.,

$$z_h(t) \circ \Phi_v(\cdot; t^{-1}) = z_h(\infty) \quad \text{on } \{|z_h(\infty)| = \rho\}.$$

- Φ is $\mathbb{Z}/2\mathbb{Z}$ -equivariant (compatible with complex conjugation), i.e.,

$$\Phi_v(\sigma_v(\cdot); t^{-1}) = \sigma(t) \circ \Phi_v(\cdot; t^{-1}).$$

In other words, for $t \gg 1$, Φ provides a smooth trivialization of the family of cores $\bigsqcup_{t^{-1} \in [0, \varepsilon)} \Sigma_v^{\text{core}}(t)$, allowing us to identify each $\Sigma_v^{\text{core}}(t)$ with the fixed reference core $\Sigma_v^{\text{core}}(\infty)$. Such a trivialization is, in general, not holomorphic. We only use the trivialization to pull back integrals of 1-forms for which we only require smoothness.

A *core path* is a family of paths

$$\lambda^{\text{core}}(t) = \Phi_v(\lambda^{\text{core}}; t^{-1}) \subset \Sigma_v^{\text{core}}(t),$$

obtained by transporting a fixed path

$$\lambda^{\text{core}} \subset \Sigma_v^{\text{core}}(\infty).$$

Note that the boundary coordinate of an endpoint of a core path is independent of t .

A *collar path* associated with an edge $e = \{h, \bar{h}\}$ is a smooth family of paths

$$\lambda^{\text{collar}}(t) \subset K_h(t) \sim K_{\bar{h}}(t), \quad t \gg 1.$$

For each endpoint $\zeta(t)$ of $\lambda^{\text{collar}}(t)$, we require that

$$(4.5) \quad \lim_{t \rightarrow \infty} -\frac{\log |z_h(t)(\zeta(t))|}{\log t}$$

exists. We denote the point of e at distance (4.5) from $v(h)$ by $\zeta^\dagger$. Note that we put no restriction on the homotopy class of a collar path inside the collar; in particular, it may wind around the collar.

A family of paths $(\gamma(t))_{t \gg 1}$ is called *degeneration-adapted* if, for all sufficiently large t , it can be written as a finite concatenation of core paths and collar paths. Whenever a core path

and a collar path are consecutive, their common endpoint lies on the corresponding boundary circle of the core. In particular, its boundary coordinate is independent of t .

The *tropicalization* of a degeneration-adapted family $(\gamma(t))_{t \gg 1}$ is the oriented path

$$\gamma^\dagger \subset \Sigma^\dagger$$

obtained by contracting every core path to the corresponding vertex and replacing every collar path by the oriented segment of the corresponding edge joining the tropical limits of its endpoints. Conversely, every oriented path in $\Sigma^\dagger$ admits a degeneration-adapted lift.

Although (4.5) may appear difficult to verify directly, in our applications it is usually automatic. The endpoints of the paths that we consider will be $\mathbb{K}$ -valued points. After passing from plumbing coordinates to the toric coordinates of Section 4.1, the limit in (4.5) can equivalently be computed as $\text{val}(z_{v(h)}(\zeta(t)))$. This determines the embedded tropical limit $\pi(\zeta^\dagger)$, while the plumbing collar containing $\zeta(t)$ determines the edge of the abstract tropical curve containing $\zeta^\dagger$. Together, these two pieces of data determine $\zeta^\dagger$.

4.4. Degeneration-adapted symplectic basis. We construct a symplectic basis of $H_1(\Sigma(t), \mathbb{Z})$ that is adapted to the degeneration $\Sigma(t) \rightarrow \Sigma(\infty)$. For every vertex $v \in V(\Sigma^\dagger)$, let

$$(\mathbf{A}_v^{\text{ov}} = (A_{v,1}^{\text{ov}}, \dots, A_{v,g_v}^{\text{ov}}), \mathbf{B}_v^{\text{ov}} = (B_{v,1}^{\text{ov}}, \dots, B_{v,g_v}^{\text{ov}}))$$

be a symplectic basis of $H_1(\Sigma_v, \mathbb{Z})$ as in Section 2.7, represented by cycles contained in Σ_v^{core} . For t sufficiently large, we regard these cycles as cycles on $\Sigma(t)$ via the trivialization Φ and denote them by $A_{v,a}^{\text{ov}}(t), B_{v,a}^{\text{ov}}(t), 1 \leq a \leq g_v$. Since the trivialization is compatible with complex conjugation and since the ovals of a Harnack curve do not meet, $B_{v,a}^{\text{ov}}(t)$ is just the corresponding oval of $\Sigma_v(t)$ and the A -cycles $A_{v,a}^{\text{ov}}(t)$ are anti-invariant.

Denote

$$(4.6) \quad \mathbf{A}^{\text{comp}}(t) = (\mathbf{A}_v^{\text{ov}}(t))_{v \in V(\Sigma^\dagger)}, \quad \mathbf{B}^{\text{comp}}(t) = (\mathbf{B}_v^{\text{ov}}(t))_{v \in V(\Sigma^\dagger)}$$

Since

$$A_{v,a}^{\text{ov}} \wedge B_{v,0}^{\text{ov}} = -1,$$

we can assume they intersect transversely exactly once.

To construct the remaining cycles, we fix a spanning tree $T \subset \Sigma^\dagger$. For each edge $e \notin E(T)$, let $B_e^\dagger \subset \Sigma^\dagger$ be the clockwise-oriented cycle consisting of e together with the unique path in T joining the two endpoints of e . The cycles

$$\mathbf{B}^\dagger := (B_e^\dagger)_{e \notin E(T)}$$

form a basis of $H_1(\Sigma^\dagger, \mathbb{Z})$. For each $e \notin E(T)$, let F_e be the set of faces in the component of the $T^* \setminus \{e^*\}$ not containing the outer face f_0 , where T^* is the planar dual tree and e^* is the edge dual to e . Then, $B_e^\dagger = \sum_{f \in F_e} B_f^{\text{ov}, \dagger}$. We therefore obtain a degeneration-adapted lift by

$$(4.7) \quad \tilde{B}_e^{\text{trop}}(t) := \sum_{f \in F_e} B_f^{\text{ov}}(t),$$

where $B_f^{\text{ov}}(t) \subset \Sigma(t)$ is the oval tropicalizing to f . Since the cycles $B_f^{\text{ov}}(t)$ and $B_{v,a}^{\text{ov}}(t)$ are ovals, they are pairwise disjoint.

For $e = \{h, \bar{h}\} \notin T$, define the *seam cycle*

$$A_e^{\text{seam}}(t) := \{|z_h(t)| = \sqrt{|q_e(t)|}\} = \{|z_{\bar{h}}(t)| = \sqrt{|q_e(t)|}\}.$$

in the plumbing collar corresponding to e . Orient $A_e^{\text{seam}}(t)$ so that

$$A_e^{\text{seam}}(t) \wedge \tilde{B}_e^{\text{trop}}(t) = 1.$$

Let

$$\mathbf{A}^{\text{seam}}(t) = (A_e^{\text{seam}})_{e \notin E(T)}, \quad \tilde{\mathbf{B}}^{\text{trop}}(t) = (\tilde{B}_e^{\text{trop}})_{e \notin E(T)}.$$

Since $B_e^\dagger$ contains no edge of $E(\Sigma^\dagger) \setminus E(T)$ other than e , we have

$$\mathbf{A}^{\text{seam}}(t) \wedge \tilde{\mathbf{B}}^{\text{trop}}(t) = I.$$

Moreover, the seam cycles are mutually disjoint, and they are disjoint from all component cycles.

It remains only to make the intersection of the tropical B -cycles with the component A -cycles equal to 0. To that end, set

$$\mathbf{B}^{\text{trop}}(t) = (B_e^{\text{trop}})_{e \notin E(T)}, \quad B_e^{\text{trop}}(t) := \tilde{B}_e^{\text{trop}}(t) - \sum_{\substack{v \in V(\Sigma^\dagger) \\ 1 \leq a \leq g_v}} (A_{v,a}^{\text{ov}}(t) \wedge \tilde{B}_e^{\text{trop}}(t)) B_{v,a}^{\text{ov}}(t).$$

The cycles

$$\mathbf{A}^{\text{deg}}(t) := (\mathbf{A}^{\text{comp}}(t), \mathbf{A}^{\text{seam}}(t)), \quad \mathbf{B}^{\text{deg}}(t) := (\mathbf{B}^{\text{comp}}(t), \mathbf{B}^{\text{trop}}(t))$$

form a symplectic basis of $H_1(\Sigma(t), \mathbb{Z})$. We call $(\mathbf{A}^{\text{deg}}(t), \mathbf{B}^{\text{deg}}(t))$ a *degeneration-adapted symplectic basis*.

5. ASYMPTOTICS UNDER DEGENERATION

5.1. Stable differentials. A *stable differential* on $\Sigma(\infty)$ is a collection

$$\Omega_\bullet = (\Omega_v)_{v \in V(\Sigma^\dagger)}$$

such that, for every vertex v , the differential Ω_v is meromorphic on Σ_v , holomorphic away from the points $\{n_h \mid h \in H_v\}$, and has at worst simple poles at these points. Moreover, for every edge $e = \{h, \bar{h}\}$, the residues satisfy

$$(5.1) \quad \text{res}_{n_h} \Omega_{v(h)} + \text{res}_{n_{\bar{h}}} \Omega_{v(\bar{h})} = 0.$$

For each $e \notin E(T)$, choose the half-edge h_e so that the seam cycle $A_e^{\text{seam}}(t)$ is positively oriented in the $z_{h_e}(t)$ -coordinate. Define

$$\mathbf{a}(\Omega_\bullet) := \left(\int_{A_{v,a}} \Omega_v \right)_{\substack{v \in V(\Sigma^\dagger) \\ 1 \leq a \leq g_v}}, \quad \mathbf{r}(\Omega_\bullet) := (\text{res}_{n_{h_e}} \Omega_{v(h_e)})_{e \notin E(T)}.$$

The *tropical 1-form associated to $\Omega_\bullet$* is defined by

$$\Omega_\bullet^\dagger(h) := \text{res}_{n_h} \Omega_{v(h)}, \quad h \in H(\Sigma^\dagger).$$

The condition (5.1) gives $\Omega_\bullet^\dagger(\bar{h}) = -\Omega_\bullet^\dagger(h)$, while the residue theorem on each component gives

$$\sum_{h \in H_v} \Omega_\bullet^\dagger(h) = 0.$$

Thus, $\Omega_\bullet^\dagger$ is a tropical 1-form on $\Sigma^\dagger$.

The following two results are stated in [HN20] under the assumption that the moduli $\mathbf{u}$ are independent of t . However, [HN20, Remark 2.4] explains that these results extend to the more general setting stated below.

Theorem 5.1 ([HN20, Theorem 1.1]). *Let $\Omega_\bullet = (\Omega_v)_{v \in V(\Sigma^\dagger)}$ be a stable differential on $\Sigma(\infty)$. Then, for $t \gg 1$, there exists a unique holomorphic differential $\Omega(t)$ on $\Sigma(t)$ satisfying*

$$\int_{\mathbf{A}^{\deg(t)}} \Omega(t) = \begin{pmatrix} \mathbf{a}(\Omega_\bullet) \\ 2\pi i \mathbf{r}(\Omega_\bullet) \end{pmatrix}.$$

Moreover, for every vertex v and every compact subset $K \subset \Sigma_v \setminus \{n_h \mid h \in H_v\}$, we have

$$\Phi_v(\cdot; t^{-1})^* \Omega(t) \rightarrow \Omega_v$$

uniformly on K .

Theorem 5.2 ([HN20, Theorem 1.2]). *For $t \gg 1$, let $\gamma(t)$ be a degeneration-adapted family of paths with tropicalization $\gamma^\dagger$. Let $\Omega_\bullet = (\Omega_v)_{v \in V(\Sigma^\dagger)}$ be a stable differential on $\Sigma(\infty)$, with associated tropical 1-form $\Omega_\bullet^\dagger$, and let $\Omega(t)$ be the family of holomorphic differentials provided by Theorem 5.1. Then, as $t \rightarrow \infty$,*

$$\int_{\gamma(t)} \Omega(t) = - \left(\int_{\gamma^\dagger} \Omega_\bullet^\dagger \right) \log t + O(1).$$

5.2. The period matrix and the Abel map. The period-matrix asymptotics in this section are essentially due to Iwao [Iwa10, Theorem 4.3.1], and his analysis also gives the corresponding Abel-map asymptotics in the plane-curve setting. We rederive these statements using plumbing since it allows us to remove certain technical assumptions that he uses. The same method will also be used for normalized differentials of the third kind where Iwao's results do not apply. We first apply Theorem 5.1 to construct the basis of holomorphic differentials on $\Sigma(t)$ dual to $\mathbf{A}^{\deg(t)}$ and then apply Theorem 5.2 to this basis.

For every vertex $v \in V(\Sigma^\dagger)$, let $\omega_v^{\text{ov}} = (\omega_{v,1}^{\text{ov}}, \dots, \omega_{v,g_v}^{\text{ov}})$ be the normalized basis of holomorphic differentials on Σ_v , so that

$$\int_{\mathbf{A}_v^{\text{ov}}} \omega_v^{\text{ov}} = 2\pi i I.$$

For each vertex v and each $1 \leq a \leq g_v$, define a stable differential $\Omega_{v,a;\bullet} = (\Omega_{v,a;u})_{u \in V(\Sigma^\dagger)}$ by

$$\Omega_{v,a;u} := \begin{cases} \omega_{v,a}^{\text{ov}}, & u = v, \\ 0, & u \neq v. \end{cases}$$

Then,

$$\mathbf{a}(\Omega_{v,a;\bullet}) = 2\pi i \mathbf{e}_{v,a}, \quad \mathbf{r}(\Omega_{v,a;\bullet}) = 0,$$

where $\mathbf{e}_{v,a} \in \mathbb{C}^{g^{\text{comp}}}$ denotes the standard basis vector corresponding to (v, a) . By Theorem 5.1, there exists a unique holomorphic differential $\omega_{v,a}^{\deg}(t)$ on $\Sigma(t)$ satisfying

$$\int_{\mathbf{A}^{\deg(t)}} \omega_{v,a}^{\deg}(t) = 2\pi i \begin{pmatrix} \mathbf{e}_{v,a} \\ 0 \end{pmatrix}.$$

We next construct the differentials corresponding to the seam cycles. Let $\omega_e^{\deg,\dagger}$ be the tropical 1-form given by unit flow along $B_e^\dagger$. The forms

$$\omega^{\deg,\dagger} := (\omega_e^{\deg,\dagger})_{e \notin E(T)}$$

form a basis of tropical 1-forms on $\Sigma^\dagger$.

For each $e \notin E(T)$ and each $v \in V(\Sigma^\dagger)$, let $\Omega_{e;v}^{\deg}$ be the unique A -normalized differential of the third kind on Σ_v satisfying

$$\text{res}_{n_h} \Omega_{e;v}^{\deg} = \omega_e^{\deg, \dagger}(h), \quad h \in H_v.$$

Then,

$$\Omega_{e; \bullet}^{\deg} := (\Omega_{e;v}^{\deg})_{v \in V(\Sigma^\dagger)}$$

is a stable differential on $\Sigma(\infty)$, whose associated tropical 1-form is $\omega_e^{\deg, \dagger}$. Moreover,

$$\mathbf{a}(\Omega_{e; \bullet}^{\deg}) = 0, \quad \mathbf{r}(\Omega_{e; \bullet}^{\deg}) = \mathbf{e}_e,$$

where $\mathbf{e}_e \in \mathbb{C}^{g^\dagger}$ is the standard basis vector corresponding to e . By Theorem 5.1, there exists a unique holomorphic differential $\omega_e^{\deg}(t)$ on $\Sigma(t)$ satisfying

$$\int_{\mathbf{A}^{\deg}(t)} \omega_e^{\deg}(t) = 2\pi i \begin{pmatrix} 0 \\ \mathbf{e}_e \end{pmatrix}.$$

Therefore,

$$\boldsymbol{\omega}^{\deg}(t) := \left((\omega_{v,a}^{\deg}(t))_{\substack{v \in V(\Sigma^\dagger), \\ 1 \leq a \leq g_v}}, (\omega_e^{\deg}(t))_{e \notin E(T)} \right)$$

is the basis of holomorphic differentials on $\Sigma(t)$ normalized with respect to $\mathbf{A}^{\deg}(t)$.

Define the *degeneration-adapted tropical period matrix* by

$$\Pi^{\deg, \dagger} := \left(\int_{B_{e'}^\dagger} \omega_e^{\deg, \dagger} \right)_{e, e' \notin E(T)}.$$

By Theorem 5.2, we obtain:

Proposition 5.3. *As $t \rightarrow \infty$, the period matrix*

$$(5.2) \quad \Pi^{\deg}(t) = - \begin{pmatrix} 0 & 0 \\ 0 & \Pi^{\deg, \dagger} \end{pmatrix} \log t + O(1).$$

Similarly, for the Abel map, we have:

Proposition 5.4. *Let $(\gamma(t))_{t \gg 1}$ be a degeneration-adapted family of paths with tropicalization $\gamma^\dagger$. Then, as $t \rightarrow \infty$,*

$$\int_{\gamma(t)} \boldsymbol{\omega}^{\deg}(t) = - \begin{pmatrix} 0 \\ \int_{\gamma^\dagger} \boldsymbol{\omega}^{\deg, \dagger} \end{pmatrix} \log t + O(1).$$

In particular, suppose that $\gamma(t)$ joins the basepoint $\zeta_0(t)$ to a family of points $\zeta(t) \in \Sigma(t)$, and let $\zeta^\dagger \in \Sigma^\dagger$ be the endpoint of $\gamma^\dagger$. Then, the lift of the Abel map determined by $\gamma(t)$ satisfies

$$\mu_{\zeta_0(t)}^{\deg}(\zeta(t)) = - \begin{pmatrix} 0 \\ \int_{\zeta_0^\dagger}^{\zeta^\dagger} \boldsymbol{\omega}^{\deg, \dagger} \end{pmatrix} \log t + O(1).$$

5.3. Differentials of the third kind. The goal of this section is to obtain asymptotics for integrals of A -normalized differentials of the third kind of the form $\omega_{\beta(t)-\alpha(t)}^{\deg}(t)$, where $\alpha(t)$ and $\beta(t)$ are families of points on $\Sigma(t)$. The main idea is to regard this differential as the limit of a holomorphic differential after introducing an additional node identifying $\alpha(t)$ and $\beta(t)$. We then apply the multivariable versions of Theorem 5.1 and Theorem 5.2 proved in [HN20].

Related asymptotics for imaginary-normalized differentials of the third kind using Schottky-uniformized degenerations were obtained by Ichikawa [Ich26b, Theorem 4.7].

Proposition 5.5. *Let $\alpha \in \Sigma_{v_\alpha}^{\text{core}}(\infty)$ and $\beta \in \Sigma_{v_\beta}^{\text{core}}(\infty)$, and let*

$$\alpha(t) = \Phi_{v_\alpha}(\alpha; t^{-1}), \quad \beta(t) = \Phi_{v_\beta}(\beta; t^{-1})$$

be the corresponding points of $\Sigma(t)$. Set $\alpha^\dagger := v_\alpha$ and $\beta^\dagger := v_\beta$. Let $\delta^\dagger$ be the unique path in the spanning tree T from $\alpha^\dagger$ to $\beta^\dagger$, and define the tropical third-kind differential $\omega_{\beta^\dagger-\alpha^\dagger}^{\deg, \dagger}$ as the tropical 1-form given by unit flow along $\delta^\dagger$.

Let $(\gamma(t))_{t \gg 1}$ be a degeneration-adapted family of paths such that $\gamma(t)$ is disjoint from $\alpha(t)$ and $\beta(t)$, and let $\gamma^\dagger$ be its tropicalization. Then, as $t \rightarrow \infty$,

$$\int_{\gamma(t)} \omega_{\beta(t)-\alpha(t)}^{\deg}(t) = - \left(\int_{\gamma^\dagger} \omega_{\beta^\dagger-\alpha^\dagger}^{\deg, \dagger} \right) \log t + O(1).$$

Proof. Form a new nodal curve by identifying α and β , and let $\widehat{\Sigma}^\dagger$ be the corresponding abstract tropical curve obtained from $\Sigma^\dagger$ by adding an edge e_* between $\alpha^\dagger$ and $\beta^\dagger$.

Choose local holomorphic coordinates $z_\alpha(t)$ and $z_\beta(t)$ centered at α and β , respectively, whose coordinate disks are contained in the corresponding reference cores. Introduce an independent plumbing parameter q_* at the new node, with plumbing equation

$$z_\alpha(t)z_\beta(t) = q_*.$$

Together with the original plumbing parameters $\mathbf{q}(t)$, this gives a two-parameter family which we denote by $\widehat{\Sigma}(q_*, t)$. For $q_* \neq 0$, the additional node is smoothed, while at $q_* = 0$ its normalization is $\Sigma(t)$ with the two preimages of the node equal to $\alpha(t)$ and $\beta(t)$.

After choosing the coordinate disks sufficiently small, the complement of these disks is canonically identified with the corresponding subset of $\Sigma(t)$:

$$(5.3) \quad \widehat{\Sigma}(q_*, t) \setminus (\{|z_\alpha(t)| < \rho\} \cup \{|z_\beta(t)| < \rho\}) = \Sigma(t) \setminus (\{|z_\alpha(t)| < \rho\} \cup \{|z_\beta(t)| < \rho\}).$$

Let $\widehat{\omega}_*^\dagger$ be the tropical 1-form on $\widehat{\Sigma}^\dagger$ whose restriction to $\Sigma^\dagger$ is $\omega_{\beta^\dagger-\alpha^\dagger}^\dagger$ and whose value on e_* is chosen so that the balancing condition holds at $\alpha^\dagger$ and $\beta^\dagger$. Let $\widehat{\Omega}_{\bullet,*} = (\widehat{\Omega}_{v,*})_v$ be the stable differential associated to $\widehat{\omega}_*^\dagger$. Thus, its residues at the old nodes are given by

$$\text{res}_{n_h} \widehat{\Omega}_{v(h),*} = \widehat{\omega}_*^\dagger(h),$$

while at the two branches of the new node they are

$$\text{res}_\beta \widehat{\Omega}_{v_\beta,*} = 1, \quad \text{res}_\alpha \widehat{\Omega}_{v_\alpha,*} = -1.$$

On each component, $\widehat{\Omega}_{v,*}$ is normalized to have vanishing periods along the component A -cycles.

By the multivariable version of Theorem 5.1, there is a family of holomorphic differentials $\widehat{\omega}_*(q_*, t)$ on $\widehat{\Sigma}(q_*, t)$ with the prescribed periods which converges to $\widehat{\Omega}_{\bullet,*}$ on compact subsets away from the nodes.

Now fix t and let $q_* \rightarrow 0$. On the normalization $\Sigma(t)$, the limiting differential has simple poles at $\alpha(t)$ and $\beta(t)$, with residues -1 and 1 , respectively, and has vanishing periods along $\mathbf{A}^{\deg}(t)$. Hence, by uniqueness,

$$\lim_{q_* \rightarrow 0} \widehat{\omega}_*(q_*, t) = \omega_{\beta(t)-\alpha(t)}^{\deg}(t)$$

uniformly on compact subsets of $\Sigma(t) \setminus \{\alpha(t), \beta(t)\}$.

We now apply the multivariable version of Theorem 5.2. Since $\gamma(t)$ is degeneration-adapted and disjoint from $\alpha(t)$ and $\beta(t)$, the coordinate disks may be chosen so that $\gamma(t)$ lies in the common region (5.3). In particular, $\gamma(t)$ does not cross the new plumbing collar, so the asymptotic expansion of $\int_{\gamma(t)} \widehat{\omega}_*(q_*, t)$ has no logarithmic contribution from q_* . Letting $q_* \rightarrow 0$ therefore gives

$$\int_{\gamma(t)} \omega_{\beta(t)-\alpha(t)}^{\deg}(t) = - \left(\int_{\gamma^\dagger} \widehat{\omega}_*^\dagger \right) \log t + O(1).$$

Finally, since $\gamma^\dagger$ is contained in the original graph $\Sigma^\dagger$ and $\widehat{\omega}_*^\dagger$ restricts to $\omega_{\beta^\dagger-\alpha^\dagger}^\dagger$, we have

$$\int_{\gamma^\dagger} \widehat{\omega}_*^\dagger = \int_{\gamma^\dagger} \omega_{\beta^\dagger-\alpha^\dagger}^\dagger.$$

This proves the proposition. $\square$

5.4. The vector of Riemann constants. In this section, we compute the asymptotics of the vector of Riemann constants.

Recall that we fixed a spanning tree $T \subset \Sigma^\dagger$. For each $e \notin E(T)$, recall that the half-edge h_e was chosen so that the seam cycle $A_e^{\text{seam}}(t)$ is positively oriented in the $z_{h_e}(t)$ -coordinate. With our orientation conventions, $B_e^\dagger$ traverses e from $v(h_e)$ to $v(\bar{h}_e)$.

Let s_e be the midpoint of e . We define the *cut tropical curve* $\Sigma_{\text{cut}}^\dagger$ by cutting every edge $e \notin E(T)$ at s_e . Thus, s_e is replaced by two points s_e^- and s_e^+ , where s_e^- belongs to the half-edge incident to $v(h_e)$ and s_e^+ belongs to the half-edge incident to $v(\bar{h}_e)$. In particular, the oriented cycle $B_e^\dagger$ traverses the seam from the s_e^- -side to the s_e^+ -side.

Since T is a spanning tree, $\Sigma_{\text{cut}}^\dagger$ is itself a tree. In particular, any two points of $\Sigma_{\text{cut}}^\dagger$ are joined by a unique path. All tropical integrals in this section are taken along these unique paths.

Proposition 5.6. *Let $\Delta^{\deg}(t) = (\Delta_j^{\deg}(t))_{j=1}^g \in \mathbb{C}^g$ denote the vector of Riemann constants of $\Sigma(t)$ with respect to the symplectic basis $(\mathbf{A}^{\deg}(t), \mathbf{B}^{\deg}(t))$ and basepoint $\zeta_0(t)$. Then,*

$$\Delta^{\deg}(t) = - \left(\begin{array}{c} 0 \\ \Delta_{\deg, \dagger}^{\deg} \end{array} \right) \log t + O(1),$$

where $\Delta^{\deg, \dagger} = (\Delta_e^{\deg, \dagger})_{e \notin E(T)} \in \mathbb{R}^{g^\dagger}$ is given by

$$(5.4) \quad \Delta_e^{\deg, \dagger} = -\frac{1}{2} \Pi_{ee}^{\deg, \dagger} + \sum_{v \in V(\Sigma^\dagger)} g_v \int_{\zeta_0^\dagger}^v \omega_e^{\deg, \dagger} + \sum_{\substack{e' \neq e \\ e' \notin E(T)}} \int_{\zeta_0^\dagger}^{s_{e'}^+} \omega_e^{\deg, \dagger}.$$

Example 5.7. Consider again the abstract tropical curve in Figure 8(B), and choose the spanning tree

$$T = \{e_1, e_2, e_3\}.$$

Thus, the edges not contained in T are e_4 and e_5 . We orient the edges by

$$e_1 : v_4 \rightarrow v_1, \quad e_2 : v_3 \rightarrow v_4, \quad e_3 : v_2 \rightarrow v_3, \quad e_4, e_5 : v_1 \rightarrow v_2.$$

With our clockwise orientation convention for the bounded faces,

$$B_1^{\text{ov},\dagger} = e_5 - e_4, \quad B_2^{\text{ov},\dagger} = e_1 + e_2 + e_3 + e_4.$$

The degeneration-adapted tropical cycles are given by

$$B_{e_4}^\dagger = e_1 + e_2 + e_3 + e_4 = B_2^{\text{ov},\dagger}, \quad B_{e_5}^\dagger = e_1 + e_2 + e_3 + e_5 = B_1^{\text{ov},\dagger} + B_2^{\text{ov},\dagger}.$$

The degeneration-adapted period matrix is

$$\Pi^{\text{deg},\dagger} = \begin{pmatrix} 21 & 18 \\ 18 & 21 \end{pmatrix}.$$

We choose the tropical basepoint $\zeta_0^\dagger = v_1$. Since $B_{e_4}^\dagger$ (resp. $B_{e_5}^\dagger$) traverses e_4 (resp. e_5) from v_1 to v_2 , the point $s_{e_4}^+$ (resp. $s_{e_5}^+$) lies on the half-edge of $\Sigma_{\text{cut}}^\dagger$ incident to v_2 .

First consider $e = e_4$. Since $\omega_{e_4}^{\text{deg},\dagger}$ vanishes on e_5 , we have

$$\int_{v_1}^{s_{e_5}^+} \omega_{e_4}^{\text{deg},\dagger} = -(5 + 8 + 5) = -18.$$

Using $\Pi_{e_4 e_4}^{\text{deg},\dagger} = 21$ and evaluating (5.4), we get

$$\Delta_{e_4}^{\text{deg},\dagger} = -\frac{1}{2} \cdot 21 - 18 = -\frac{57}{2}.$$

By symmetry under exchanging e_4 and e_5 , we get

$$\Delta^{\text{deg},\dagger} = \begin{pmatrix} -57/2 \\ -57/2 \end{pmatrix}.$$

The rest of this section is devoted to the proof of Proposition 5.6.

Let Σ be a compact Riemann surface of genus g , let $\zeta_0 \in \Sigma$ be a basepoint, and let $(\mathbf{A}, \mathbf{B})$ be a symplectic basis represented by a canonical system of loops based at ζ_0 . Let ω be the corresponding normalized basis of holomorphic differentials, let Π be the period matrix, and let Δ be the vector of Riemann constants.

Dissecting Σ along $(\mathbf{A}, \mathbf{B})$ gives a polygon $\mathfrak{P}$. Each A -cycle A_j has two lifts to $\mathfrak{P}$; label them A_j^- and A_j^+ so that the corresponding B -cycle crosses from A_j^- to A_j^+ . Then, Fay [Fay73, Eq. (13)] gives

$$(5.5) \quad \Delta_j = \pi i - \frac{1}{2} \Pi_{jj} + \frac{1}{2\pi i} \sum_{k \neq j} \int_{\zeta \in A_k^+} \omega_k(\zeta) \int_{\zeta_0}^{\zeta} \omega_j, \quad 1 \leq j \leq g,$$

where the inner integral is taken along the unique path from ζ_0 to ζ contained in $\mathfrak{P}$.

In order to apply (5.5) to $\Sigma(t)$, we first choose representatives of the degeneration-adapted symplectic basis that pass through the basepoint $\zeta_0(t)$. We use the component A -cycles $A_{v,a}^{\text{ov}}(t)$ and the seam cycles $A_e^{\text{seam}}(t)$ themselves, without homotoping the latter out of the plumbing collars.

For each $A_k^{\deg}(t)$, choose a point $\eta_k(t) \in A_k^{\deg}(t)$. Define

$$\tau_k^\dagger := \begin{cases} v, & k = (v, a), \\ s_e^+, & k = e \notin E(T). \end{cases}$$

Let $\lambda_k^\dagger$ be the unique path in $\Sigma_{\text{cut}}^\dagger$ from $\zeta_0^\dagger$ to $\tau_k^\dagger$, and choose a degeneration-adapted lift $\lambda_k(t)$ joining $\zeta_0(t)$ to $\eta_k(t)$. In the case $k = e \notin E(T)$, we choose $\lambda_k(t)$ to approach the seam from the side corresponding to s_e^+ . We may then replace $A_k^{\deg}(t)$ by

$$\widehat{A}_k^{\deg}(t) = \lambda_k(t) \cdot A_k^{\deg}(t) \cdot \lambda_k(t)^{-1}.$$

After a small perturbation, we may replace $\widehat{A}_k^{\deg}(t)$ by a simple loop based at $\zeta_0(t)$. Similarly, we choose representatives $\widehat{B}_k^{\deg}(t)$ of the B -cycles passing through $\zeta_0(t)$ so that

$$(\widehat{\mathbf{A}}^{\deg}(t), \widehat{\mathbf{B}}^{\deg}(t))$$

forms a canonical system of loops based at $\zeta_0(t)$. Dissecting $\Sigma(t)$ along these loops, we obtain a polygon $\mathfrak{P}_{\Sigma(t)}^{\deg}$. Under tropicalization, the two copies of $A_e^{\text{seam}}(t)$ on the boundary of this polygon correspond to s_e^- and s_e^+ , with $(A_e^{\text{seam}}(t))^+$ corresponding to s_e^+ .

Index the degeneration-adapted A -cycles by $\{1, \dots, g\}$ so that the indices corresponding to $\{(v, a) \mid v \in V(\Sigma^\dagger), 1 \leq a \leq g_v\}$ appear first and followed by the indices corresponding to $E(\Sigma^\dagger) \setminus E(T)$. Applying (5.5), we obtain

$$\Delta_j^{\deg}(t) = \pi i - \frac{1}{2} \Pi_{jj}^{\deg}(t) + \frac{1}{2\pi i} \sum_{k \neq j} \int_{\zeta \in (\widehat{A}_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t).$$

We now show that we may replace $\widehat{A}_k^{\deg}(t)$ by $A_k^{\deg}(t)$ in this formula.

Lemma 5.8. *For $1 \leq j \leq g$, we have*

$$(5.6) \quad \Delta_j^{\deg}(t) = \pi i - \frac{1}{2} \Pi_{jj}^{\deg}(t) + \frac{1}{2\pi i} \sum_{k \neq j} \int_{\zeta \in (A_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t).$$

Proof. By A -normalization of $\omega^{\deg}(t)$, for $k \neq j$, after traversing $A_k^{\deg}(t)$ the function

$$\zeta \mapsto \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t)$$

changes by

$$\int_{A_k^{\deg}(t)} \omega_j^{\deg}(t) = 2\pi i \delta_{kj} = 0.$$

Therefore, this function has the same value on the two copies of $\lambda_k(t)$, and their contributions cancel. $\square$

The following is obtained by summing [HN20, Eq. (3.11)] over all k and using [HN20, Lemma 3.2].

Lemma 5.9. *Let $\Omega_\bullet = (\Omega_v)_{v \in V(\Sigma^\dagger)}$ be a stable differential on $\Sigma(\infty)$, with associated tropical 1-form $\Omega_\bullet^\dagger$, and let $\Omega(t)$ be the family of holomorphic differentials on $\Sigma(t)$ provided by Theorem 5.1. For $e \in E(\Sigma^\dagger) \setminus E(T)$, we have*

$$\Omega(t) = \left(\frac{\Omega_\bullet^\dagger(h_e)}{z_{h_e}} + O(1) \right) dz_{h_e}$$

uniformly on $A_e^{\text{seam}}(t)$.

Define

$$\omega_j^{\text{deg}, \dagger} := \begin{cases} 0, & j = (v, a), \\ \omega_e^{\text{deg}, \dagger}, & j = e \notin E(T). \end{cases}$$

Lemma 5.10. *Let $1 \leq j, k \leq g$ with $j \neq k$. Then, uniformly for $\zeta \in (A_k^{\text{deg}}(t))^+$,*

$$(5.7) \quad \int_{\zeta_0(t)}^{\zeta} \omega_j^{\text{deg}}(t) = - \left(\int_{\zeta_0^\dagger}^{\tau_k^\dagger} \omega_j^{\text{deg}, \dagger} \right) \log t + O(1).$$

Moreover,

$$\int_{(A_k^{\text{deg}}(t))^+} \left| \omega_k^{\text{deg}}(t) \right| = O(1).$$

Consequently,

$$\frac{1}{2\pi i} \int_{\zeta \in (A_k^{\text{deg}}(t))^+} \omega_k^{\text{deg}}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_j^{\text{deg}}(t) = - \left(\int_{\zeta_0^\dagger}^{\tau_k^\dagger} \omega_j^{\text{deg}, \dagger} \right) \log t + O(1).$$

Proof. Choose $\eta_k(t) \in (A_k^{\text{deg}}(t))^+$ as above and choose a path in $\mathfrak{P}_{\Sigma(t)}^{\text{deg}}$ from $\zeta_0(t)$ to $\eta_k(t)$ that is a degeneration-adapted lift of the unique path in $\Sigma_{\text{cut}}^\dagger$ from $\zeta_0^\dagger$ to $\tau_k^\dagger$. By Theorem 5.2,

$$(5.8) \quad \int_{\zeta_0(t)}^{\eta_k(t)} \omega_j^{\text{deg}}(t) = - \left(\int_{\zeta_0^\dagger}^{\tau_k^\dagger} \omega_j^{\text{deg}, \dagger} \right) \log t + O(1).$$

Suppose first that $k = (v, a)$. Then $A_k^{\text{deg}}(t) = A_{v,a}^{\text{ov}}(t)$ is contained in a component core, so for $\zeta \in (A_k^{\text{deg}}(t))^+$, the path from $\eta_k(t)$ to ζ may be chosen inside the same component core. By Theorem 5.1, after identifying the cores using the trivialization Φ , the differentials $\omega_j^{\text{deg}}(t)$ converge uniformly on a compact neighborhood of $A_k^{\text{deg}}(t)$. Therefore,

$$(5.9) \quad \int_{\eta_k(t)}^{\zeta} \omega_j^{\text{deg}}(t) = O(1)$$

uniformly in ζ .

Now suppose that $k = e \notin E(T)$. The points $\eta_k(t)$ and ζ lie on the same copy $(A_e^{\text{seam}}(t))^+$ of the seam. By Lemma 5.9, the integral of $\omega_j^{\text{deg}}(t)$ along any subarc of the seam is uniformly bounded. Hence, again,

$$(5.10) \quad \int_{\eta_k(t)}^{\zeta} \omega_j^{\text{deg}}(t) = O(1)$$

uniformly in ζ . Combining this with (5.8) gives (5.7).

For $k = (v, a)$, the same uniform convergence on the component core, applied to $\omega_k^{\text{deg}}(t)$, gives

$$\int_{(A_k^{\text{deg}}(t))^+} \left| \omega_k^{\text{deg}}(t) \right| = O(1).$$

For $k = e \notin E(T)$, the same estimate follows from Lemma 5.9. This proves the first two assertions.

By the first two assertions, there exist constants $M_1, M_2 > 0$ such that, for all sufficiently large t ,

$$\frac{1}{2\pi} \int_{(A_k^{\deg}(t))^+} \left| \omega_k^{\deg}(t) \right| \leq M_1, \quad \sup_{\zeta \in (A_k^{\deg}(t))^+} \left| \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t) - c \log t \right| \leq M_2,$$

where $c := - \int_{\zeta_0^\dagger}^{\tau_k^\dagger} \omega_j^{\deg, \dagger}$. Using the normalization

$$\int_{(A_k^{\deg}(t))^+} \omega_k^{\deg}(t) = 2\pi i,$$

we obtain

$$\begin{aligned} & \left| \frac{1}{2\pi i} \int_{\zeta \in (A_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t) - c \log t \right| \\ &= \left| \frac{1}{2\pi i} \int_{\zeta \in (A_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \left(\int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t) - c \log t \right) \right| \\ &\leq \frac{1}{2\pi} \int_{(A_k^{\deg}(t))^+} \left| \omega_k^{\deg}(t) \right| \sup_{\zeta \in (A_k^{\deg}(t))^+} \left| \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t) - c \log t \right| \\ &\leq M_1 M_2. \end{aligned}$$

Therefore,

$$\frac{1}{2\pi i} \int_{\zeta \in (A_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_j^{\deg}(t) = - \left(\int_{\zeta_0^\dagger}^{\tau_k^\dagger} \omega_j^{\deg, \dagger} \right) \log t + O(1).$$

□

Proof of Proposition 5.6. Suppose first that $j = (v, a)$. The stable differential associated to $\omega_{v,a}^{\deg}(t)$ has zero associated tropical 1-form. Hence, by Lemma 5.10, for every $k \neq (v, a)$,

$$\frac{1}{2\pi i} \int_{\zeta \in (A_k^{\deg}(t))^+} \omega_k^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_{v,a}^{\deg}(t) = O(1).$$

Moreover, by Proposition 5.3, $\Pi_{(v,a),(v,a)}^{\deg}(t) = O(1)$, and therefore,

$$\Delta_{v,a}^{\deg}(t) = O(1).$$

Now let $j = e \notin E(T)$. By Proposition 5.3,

$$\Pi_{ee}^{\deg}(t) = -\Pi_{ee}^{\deg, \dagger} \log t + O(1).$$

For a component index $k = (v, a)$, Lemma 5.10 gives

$$\frac{1}{2\pi i} \int_{\zeta \in (A_{v,a}^{\deg}(t))^+} \omega_{v,a}^{\deg}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_e^{\deg}(t) = - \left(\int_{\zeta_0^\dagger}^v \omega_e^{\deg, \dagger} \right) \log t + O(1).$$

Summing over $1 \leq a \leq g_v$ and then over $v \in V(\Sigma^\dagger)$ gives

$$- \sum_{v \in V(\Sigma^\dagger)} g_v \left(\int_{\zeta_0^\dagger}^v \omega_e^{\deg, \dagger} \right) \log t + O(1).$$

Next suppose that $k = e' \neq e$ with $e' \notin E(T)$. Again by Lemma 5.10,

$$\frac{1}{2\pi i} \int_{\zeta \in (A_{e'}^{\text{seam}}(t))^+} \omega_{e'}^{\text{deg}}(t)(\zeta) \int_{\zeta_0(t)}^{\zeta} \omega_e^{\text{deg}}(t) = - \left(\int_{\zeta_0^\dagger}^{s_{e'}^+} \omega_e^{\text{deg}, \dagger} \right) \log t + O(1).$$

Summing over $e' \neq e$ and substituting these asymptotics into (5.6), we obtain

$$\Delta_e^{\text{deg}}(t) = \left(\frac{1}{2} \Pi_{ee}^{\text{deg}, \dagger} - \sum_{v \in V(\Sigma^\dagger)} g_v \int_{\zeta_0^\dagger}^v \omega_e^{\text{deg}, \dagger} - \sum_{\substack{e' \neq e \\ e' \notin E(T)}} \int_{\zeta_0^\dagger}^{s_{e'}^+} \omega_e^{\text{deg}, \dagger} \right) \log t + O(1).$$

Therefore,

$$\Delta^{\text{deg}}(t) = - \begin{pmatrix} 0 \\ \Delta^{\text{deg}, \dagger} \end{pmatrix} \log t + O(1).$$

□

5.5. The theta function. Following Mikhalkin–Zharkov [MZ08, Section 5.2], but using the max-plus convention, we define the *tropical theta function* associated to the tropical period matrix $\Pi^{\text{deg}, \dagger}$ by

$$\theta^\dagger(\cdot \mid \Pi^{\text{deg}, \dagger}) : \mathbb{R}^{g^\dagger} \rightarrow \mathbb{R}, \quad \theta^\dagger(\boldsymbol{\xi}^\dagger \mid \Pi^{\text{deg}, \dagger}) := \max_{\boldsymbol{l} \in \mathbb{Z}^{g^\dagger}} \left(-\frac{1}{2} \langle \boldsymbol{l}, \Pi^{\text{deg}, \dagger} \boldsymbol{l} \rangle - \langle \boldsymbol{l}, \boldsymbol{\xi}^\dagger \rangle \right).$$

Proposition 5.11. *Suppose that $\boldsymbol{\xi}(t) \in \mathbb{R}^g$ and that*

$$(5.11) \quad \boldsymbol{\xi}(t) = - \begin{pmatrix} 0 \\ \boldsymbol{\xi}^\dagger \end{pmatrix} \log t + O(1)$$

for some $\boldsymbol{\xi}^\dagger \in \mathbb{R}^{g^\dagger}$. Then,

$$\log \theta(\boldsymbol{\xi}(t) \mid \Pi^{\text{deg}}(t)) = \theta^\dagger(\boldsymbol{\xi}^\dagger \mid \Pi^{\text{deg}, \dagger}) \log t + O(1).$$

Proof. Write

$$\boldsymbol{m} = \begin{pmatrix} \boldsymbol{k} \\ \boldsymbol{l} \end{pmatrix} \in \mathbb{Z}^{g^{\text{comp}}} \times \mathbb{Z}^{g^\dagger}.$$

By definition,

$$(5.12) \quad \theta(\boldsymbol{\xi}(t) \mid \Pi^{\text{deg}}(t)) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp \left(\frac{1}{2} \langle \boldsymbol{m}, \Pi^{\text{deg}}(t) \boldsymbol{m} \rangle + \langle \boldsymbol{m}, \boldsymbol{\xi}(t) \rangle \right).$$

Using (5.2) and (5.11), we obtain

$$\frac{1}{2} \langle \boldsymbol{m}, \Pi^{\text{deg}}(t) \boldsymbol{m} \rangle + \langle \boldsymbol{m}, \boldsymbol{\xi}(t) \rangle = \left(-\frac{1}{2} \langle \boldsymbol{l}, \Pi^{\text{deg}, \dagger} \boldsymbol{l} \rangle - \langle \boldsymbol{l}, \boldsymbol{\xi}^\dagger \rangle \right) \log t + O(1).$$

Since $\boldsymbol{\xi}(t)$ and $\Pi^{\text{deg}}(t)$ are real, all summands in (5.12) are positive. In particular, there is no cancellation of leading terms. Therefore,

$$\begin{aligned} \log \theta(\boldsymbol{\xi}(t) \mid \Pi^{\text{deg}}(t)) &= \max_{\boldsymbol{l} \in \mathbb{Z}^{g^\dagger}} \left(-\frac{1}{2} \langle \boldsymbol{l}, \Pi^{\text{deg}, \dagger} \boldsymbol{l} \rangle - \langle \boldsymbol{l}, \boldsymbol{\xi}^\dagger \rangle \right) \log t + O(1) \\ &= \theta^\dagger(\boldsymbol{\xi}^\dagger \mid \Pi^{\text{deg}, \dagger}) \log t + O(1). \end{aligned}$$

□

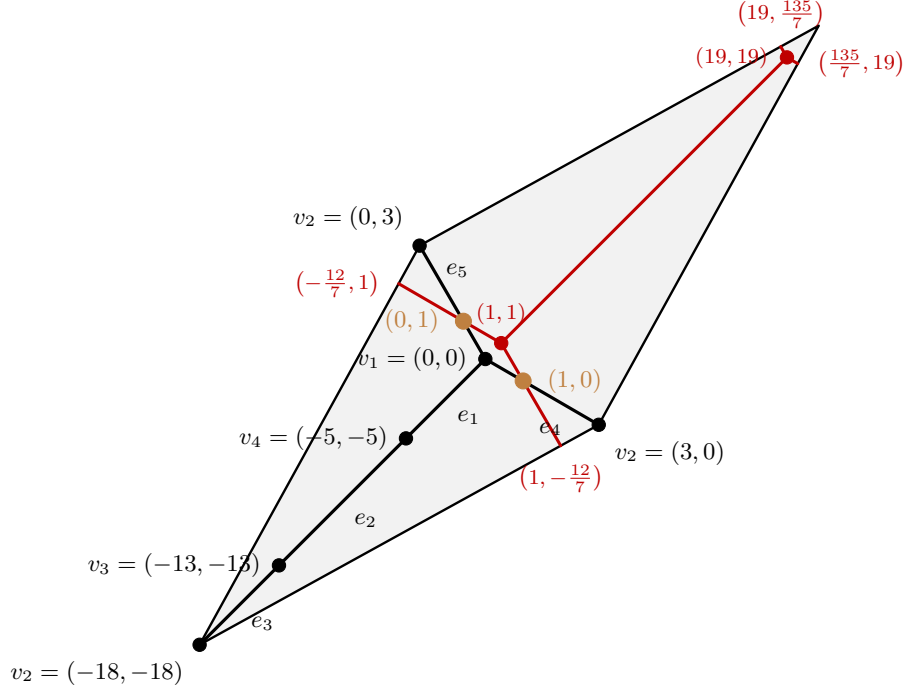


FIGURE 9. Tropical Riemann's theorem for the abstract tropical curve in Figure 8(B). We take $\zeta_0^\dagger = v_1$ and $\xi^\dagger = (1, 1)$. The parallelogram is a translated fundamental domain for $\text{Jac}(\Sigma^\dagger)$ whose side vectors are the columns of $\Pi^{\text{deg},\dagger}$. For clarity, the picture has been linearly rescaled. The black graph is the Abel image $\mu_{v_1}^{\text{deg},\dagger}(\Sigma^\dagger)$ and the red graph is the translated tropical theta divisor $\Theta^\dagger + \xi^\dagger - \Delta^{\text{deg},\dagger}$. Their intersection consists of the two brown points $(1, 0)$ and $(0, 1)$.

Example 5.12. Let us verify for the abstract tropical curve in Figure 8(B) that $\Delta^{\text{deg},\dagger}$ agrees with the tropical vector of Riemann constants appearing in the following tropical version of Riemann's theorem due to Mikhalkin–Zharkov [MZ08, Theorem 6.5], formulated as in Remark 2.5. Let

$$\Theta^\dagger := \{x \in \text{Jac}(\Sigma^\dagger) \mid \theta^\dagger(x \mid \Pi^{\text{deg},\dagger}) \text{ achieves its maximum at least twice}\}$$

be the tropical theta divisor. Then, there is a unique constant $[\Delta^{\text{MZ}}] \in \text{Jac}(\Sigma^\dagger)$ such that for any $[\xi^\dagger] \in \text{Jac}(\Sigma^\dagger)$,

$$D^\dagger = \sum_{v \in V(\Sigma^\dagger)} g_v \cdot v + \mu_{\zeta_0^\dagger}^{\text{deg},\dagger}(\Sigma^\dagger) \cap_{\text{stable}} (\Theta^\dagger + [\xi^\dagger] - [\Delta^{\text{MZ}}])$$

is a divisor such that

$$\mu_{\zeta_0^\dagger}^{\text{deg},\dagger}(D^\dagger) = [\xi^\dagger].$$

Here, stable intersection means the limit of intersections with generic perturbations. We check that

$$[\Delta^{\text{deg},\dagger}] = [\Delta^{\text{MZ}}].$$

Since

$$\Pi^{\deg, \dagger} = \begin{pmatrix} 21 & 18 \\ 18 & 21 \end{pmatrix},$$

we represent the tropical Jacobian by drawing a translated fundamental parallelogram whose sides are given by the columns of $\Pi^{\deg, \dagger}$. Take $\xi^\dagger = (1, 1)$ and recall from Example 5.7 that the fixed basepoint is $\zeta_0^\dagger = v_1$ and $\Delta^{\deg, \dagger} = \begin{pmatrix} -57/2 \\ -57/2 \end{pmatrix}$. The translate appearing in tropical Riemann's theorem is

$$\Theta^\dagger + \xi^\dagger - \Delta^{\deg, \dagger},$$

which is the red graph in Figure 9. The Abel image $\mu_{v_1}^{\deg, \dagger}(\Sigma^\dagger)$ is the black graph in Figure 9, from which we get that the intersection, which is transverse and thus agrees with the stable intersection, is

$$\mu_{v_1}^{\deg, \dagger}(\Sigma^\dagger) \cap (\Theta^\dagger + \xi^\dagger - \Delta^{\deg, \dagger}) = \{(1, 0), (0, 1)\}$$

Since all vertices have genus zero, $D^\dagger$ is the preimage of these two points. Finally,

$$\mu_{v_1}^{\deg, \dagger}(D^\dagger) = (1, 0) + (0, 1) = (1, 1) = \xi^\dagger.$$

Therefore,

$$[\Delta^{\deg, \dagger}] = [\Delta^{\text{MZ}}].$$

6. TROPICALIZATION OF FOCK'S INVERSE SPECTRAL TRANSFORM

6.1. Tropical dimer model. Let

$$\text{wt}^\dagger : E(G) \rightarrow \mathbb{Q}$$

be a *tropical edge weight*. Two tropical edge weights are *tropically gauge equivalent* if there is a function

$$f : B(G) \sqcup W(G) \rightarrow \mathbb{Q}$$

such that

$$\text{wt}_2^\dagger(\mathbf{bw}) = \text{wt}_1^\dagger(\mathbf{bw}) - f(\mathbf{b}) + f(\mathbf{w}) \quad \text{for every } \mathbf{bw} \in E(G).$$

We denote the tropical gauge-equivalence class of $\text{wt}^\dagger$ by $[\text{wt}^\dagger]$.

If $\gamma = (\mathbf{b}_1 \xrightarrow{\mathbf{e}_1} \mathbf{w}_1 \xrightarrow{\mathbf{e}_2} \mathbf{b}_2 \xrightarrow{\mathbf{e}_3} \dots \xrightarrow{\mathbf{e}_{2k-1}} \mathbf{w}_k \xrightarrow{\mathbf{e}_{2k}} \mathbf{b}_1)$ is a cycle in G , define its *tropical cycle weight* by

$$[\text{wt}^\dagger](\gamma) := \sum_{j=1}^k (\text{wt}^\dagger(\mathbf{e}_{2j-1}) - \text{wt}^\dagger(\mathbf{e}_{2j})).$$

This depends only on the tropical gauge-equivalence class.

6.2. The tropical inverse spectral transform. We now define a tropical version of Fock's inverse spectral transform. A *tropical spectral datum associated to G* is a triple $(\Sigma^\dagger, D^\dagger, \nu^\dagger)$, where:

- (1) $(\Sigma^\dagger, \ell, \mathbf{g})$ is the abstract tropical curve associated to an embedded tropical spectral curve $\Sigma_{\text{emb}}^\dagger$ with Newton polygon N .
- (2) $D^\dagger$ is a tropical effective divisor of degree g .
- (3) $\nu^\dagger$ is the tropical angle map, assigning to each zig-zag path $\alpha \in Z(G)$ a vertex $\alpha^\dagger \in V(\Sigma^\dagger)$ such that $\alpha^\dagger$ is the endpoint of an unbounded edge dual to the side $E^*(\alpha)$ of N .

Fix a spanning tree $T \subset \Sigma^\dagger$ and a basepoint $\zeta_0^\dagger \in \Sigma^\dagger$, and use the corresponding degeneration-adapted tropical period matrix $\Pi^{\deg, \dagger}$, Abel map $\mu_{\zeta_0^\dagger}^{\deg, \dagger}$, and vector of Riemann constants $\Delta^{\deg, \dagger}$. Extending $\nu^\dagger$ linearly to divisors supported on $Z(G)$ gives the *tropical discrete Abel map* $\mathbf{d}^\dagger$.

For a wedge $\mathfrak{w}$ with associated α, β_-, β_+ and adjacent faces $\mathfrak{f}_-, \mathfrak{f}_+$, define its *tropical Fock wedge weight* by

$$(6.1) \quad \begin{aligned} \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}) := & \theta^\dagger \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathfrak{f}_-)) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(D^\dagger) + \Delta^{\deg, \dagger} \middle| \Pi^{\deg, \dagger} \right) \\ & - \theta^\dagger \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathfrak{f}_+)) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(D^\dagger) + \Delta^{\deg, \dagger} \middle| \Pi^{\deg, \dagger} \right) \\ & - \int_{\zeta_0^\dagger}^{\alpha^\dagger} \omega_{\beta_+^\dagger - \beta_-^\dagger}^\dagger. \end{aligned}$$

The integrals from $\zeta_0^\dagger$ to a point of $\Sigma^\dagger$ are taken along the unique path in the cut tropical $\Sigma_{\text{cut}}^\dagger$ joining the two points.

If $\gamma = \mathfrak{w}_1 + \cdots + \mathfrak{w}_r$, define the corresponding *tropical Fock cycle weight* by

$$(6.2) \quad [\text{wt}_{\text{Fock}}^\dagger](\gamma) := \sum_{j=1}^r \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}_j).$$

Thus, we have defined a map

$$\begin{aligned} (\lambda_{\text{Fock}}^{-1})^\dagger : \{\text{tropical spectral data associated to } G\} &\rightarrow \{\text{tropical edge weights on } G\}/\text{gauge} \\ ((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger) &\mapsto [\text{wt}_{\text{Fock}}^\dagger] \end{aligned}$$

6.3. Main theorem. Let $[\text{wt}](t)$ be $\mathbb{K}_{>0}$ -valued edge weights modulo gauge equivalence satisfying Assumption 4.1, and let

$$[\text{wt}^\dagger] := \text{val}([\text{wt}](t))$$

be its tropicalization. For $t \gg 1$, apply the spectral transform:

$$(6.3) \quad \lambda([\text{wt}](t)) = (\Sigma(t), D(t), \nu(t)).$$

Recall that the family of spectral curves $\Sigma(t)$ degenerates to a nodal curve $\Sigma(\infty)$, and that the corresponding embedded tropical spectral curve $\Sigma_{\text{emb}}^\dagger$ determines an abstract tropical curve $(\Sigma^\dagger, \ell, \mathbf{g})$. Fix a spanning tree $T \subset \Sigma^\dagger$ and a basepoint $\zeta_0(t) \in \Sigma(t)$ lying in the core of a component Σ_v , with tropical limit $\zeta_0^\dagger = v$. The standard divisor

$$D(t) = \sum_{j=1}^g \zeta_j(t)$$

tropicalizes to an embedded tropical divisor

$$D_{\text{emb}}^\dagger := \sum_{j=1}^g \text{val}(\zeta_j(t))$$

on $\Sigma_{\text{emb}}^\dagger$. For each j , the component core or plumbing collar containing $\zeta_j(t)$ determines the corresponding vertex or edge of $\Sigma^\dagger$. Thus, the degenerating family $D(t)$ determines a lift

$$D^\dagger \in \text{Sym}^g(\Sigma^\dagger)$$

satisfying $\pi(D^\dagger) = D_{\text{emb}}^\dagger$.

For $\alpha \in Z(G)$, write $\alpha(t) := \nu(t)(\alpha)$. The tentacle of $\Sigma^\circ(t)$ corresponding to $\alpha(t)$ tropicalizes to the unbounded edge of $\Sigma_{\text{emb}}^{\circ\dagger}$ contained in the line

$$\langle [\alpha], \mathbf{x} \rangle + C_\alpha^\dagger = 0,$$

where

$$C_\alpha^\dagger := \text{val}(C_\alpha(t))$$

is the *tropical Casimir*. Let $\alpha^\dagger$ be the vertex of $\Sigma_{\text{emb}}^\dagger$ incident to this unbounded edge, which we identify with the corresponding vertex of $\Sigma^\dagger$. This defines the *tropical angle map*

$$\nu^\dagger : Z(G) \rightarrow V(\Sigma^\dagger), \quad \alpha \mapsto \alpha^\dagger.$$

We thus obtain a tropical spectral datum $((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger)$ associated to the family (6.3).

Theorem 6.1. *Let $[\text{wt}](t)$ be $\mathbb{K}_{>0}$ -valued edge weights modulo gauge equivalence satisfying Assumption 4.1, and let $\lambda([\text{wt}](t)) = (\Sigma(t), D(t), \nu(t))$. Let $((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger)$ be the corresponding tropical spectral datum. Then,*

$$(\lambda_{\text{Fock}}^{-1})^\dagger((\Sigma^\dagger, \ell, \mathbf{g}), D^\dagger, \nu^\dagger) = [\text{wt}^\dagger].$$

In other words, tropicalization commutes with Fock's inverse spectral transform.

Proof. We first rewrite Fock's inverse spectral transform in degeneration-adapted coordinates. Recall from Section 4.4 that

$$\mathbf{B}^{\text{deg}}(t) = M_T \mathbf{B}^{\text{ov}}(t), \quad M_T \in \text{GL}_g(\mathbb{Z}).$$

We may choose the oval-adapted A -cycles to be $\mathbf{A}^{\text{ov}}(t) = M_T^\top \mathbf{A}^{\text{deg}}(t)$.

Standard change-of-basis formulas (see, for example, [Fay73]) give that the corresponding normalized holomorphic differentials and period matrices satisfy

$$\boldsymbol{\omega}^{\text{ov}}(t) = M_T^{-1} \boldsymbol{\omega}^{\text{deg}}(t), \quad \Pi^{\text{ov}}(t) = M_T^{-1} \Pi^{\text{deg}}(t) (M_T^{-1})^\top,$$

and that

$$\mu_{\zeta_0(t)}^{\text{ov}} = M_T^{-1} \mu_{\zeta_0(t)}^{\text{deg}}, \quad \boldsymbol{\xi}^{\text{ov}}(t) = M_T^{-1} \boldsymbol{\xi}^{\text{deg}}(t), \quad \boldsymbol{\Delta}^{\text{ov}}(t) = M_T^{-1} \boldsymbol{\Delta}^{\text{deg}}(t).$$

Since $M_T \in \text{GL}_g(\mathbb{Z})$, reindexing the series expansion of the theta function gives

$$\theta(M_T^{-1} \boldsymbol{\xi} | M_T^{-1} \Pi^{\text{deg}} (M_T^{-1})^\top) = \theta(\boldsymbol{\xi} | \Pi^{\text{deg}}).$$

Moreover, the normalized differential of the third kind is unchanged: vanishing of its periods along $\mathbf{A}^{\text{ov}}(t)$ is equivalent to vanishing of its periods along $\mathbf{A}^{\text{deg}}(t)$.

Let $\mathbf{w}$ be a wedge with associated zig-zag paths α, β_-, β_+ and adjacent faces $\mathbf{f}_-, \mathbf{f}_+$, as in Section 2.10. Fock's wedge weight therefore takes the same form in degeneration-adapted coordinates:

$$(6.4) \quad \begin{aligned} \text{wt}_{\text{Fock}}(t)(\mathbf{w}) = & - \frac{\theta\left(\mu_{\zeta_0(t)}^{\text{deg}}(\mathbf{d}(\mathbf{f}_-)) - \mu_{\zeta_0(t)}^{\text{deg}}(D(t)) + \boldsymbol{\Delta}^{\text{deg}}(t) \middle| \Pi^{\text{deg}}(t)\right)}{\theta\left(\mu_{\zeta_0(t)}^{\text{deg}}(\mathbf{d}(\mathbf{f}_+)) - \mu_{\zeta_0(t)}^{\text{deg}}(D(t)) + \boldsymbol{\Delta}^{\text{deg}}(t) \middle| \Pi^{\text{deg}}(t)\right)} \\ & \times \exp\left(\int_{\zeta_0(t)}^{\alpha(t)} \omega_{\beta_+(t) - \beta_-(t)}^{\text{deg}}(t)\right). \end{aligned}$$

The integrals from $\zeta_0(t)$ to any point $\zeta(t)$ are taken along degeneration-adapted lifts of the unique path in $\Sigma_{\text{cut}}^\dagger$ from $\zeta_0^\dagger$ to $\zeta^\dagger$; in particular, these paths tropicalize to the corresponding paths in (6.1).

By Proposition 5.4,

$$\begin{aligned}\mu_{\zeta_0(t)}^{\deg}(D(t)) &= - \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(D^\dagger) \right) \log t + O(1), \\ \mu_{\zeta_0(t)}^{\deg}(\mathbf{d}(\mathbf{f})) &= - \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f})) \right) \log t + O(1).\end{aligned}$$

Together with Proposition 5.6, we obtain

$$\begin{aligned}(6.5) \quad & \mu_{\zeta_0(t)}^{\deg}(\mathbf{d}(\mathbf{f})) - \mu_{\zeta_0(t)}^{\deg}(D(t)) + \Delta^{\deg}(t) \\ &= - \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f})) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(D^\dagger) + \Delta^{\deg, \dagger} \right) \log t + O(1).\end{aligned}$$

By Proposition 2.6,

$$\mu_{\zeta_0(t)}^{\text{ov}}(\mathbf{d}(\mathbf{f})) \in \mathbb{R}^g, \quad \mu_{\zeta_0(t)}^{\text{ov}}(D(t)) \in \pi i \mathbf{1} + \mathbb{R}^g, \quad \Delta^{\text{ov}}(t) \in \pi i \mathbf{1} + \mathbb{R}^g,$$

and therefore,

$$\mu_{\zeta_0(t)}^{\text{ov}}(\mathbf{d}(\mathbf{f})) - \mu_{\zeta_0(t)}^{\text{ov}}(D(t)) + \Delta^{\text{ov}}(t) \in \mathbb{R}^g.$$

Since the corresponding degeneration-adapted theta argument is obtained by multiplication by $M_T \in \text{GL}_g(\mathbb{Z})$, it also lies in $\mathbb{R}^g$. Therefore, Proposition 5.11 and (6.5) give

$$\begin{aligned}(6.6) \quad & \log \theta \left(\mu_{\zeta_0(t)}^{\deg}(\mathbf{d}(\mathbf{f})) - \mu_{\zeta_0(t)}^{\deg}(D(t)) + \Delta^{\deg}(t) \middle| \Pi^{\deg}(t) \right) \\ &= \theta^\dagger \left(\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f})) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(D^\dagger) + \Delta^{\deg, \dagger} \middle| \Pi^{\deg, \dagger} \right) \log t + O(1).\end{aligned}$$

On the other hand, Proposition 5.5 gives

$$(6.7) \quad \int_{\zeta_0(t)}^{\alpha(t)} \omega_{\beta_+(t) - \beta_-(t)}^{\deg}(t) = - \left(\int_{\zeta_0^\dagger}^{\alpha^\dagger} \omega_{\beta_+^\dagger - \beta_-^\dagger}^\dagger \right) \log t + O(1),$$

where the tropical integral is taken along the unique path in $\Sigma_{\text{cut}}^\dagger$ from $\zeta_0^\dagger$ to $\alpha^\dagger$.

Taking the logarithm of the absolute value of (6.4) and substituting (6.6) and (6.7), we obtain

$$(6.8) \quad \log |\text{wt}_{\text{Fock}}(t)(\mathbf{w})| = \text{wt}_{\text{Fock}}^\dagger(\mathbf{w}) \log t + O(1).$$

Now let $\gamma = \mathbf{w}_1 + \cdots + \mathbf{w}_r$ be a cycle in G . By Theorem 2.8,

$$[\text{wt}](t)(\gamma) = [\text{wt}_{\text{Fock}}](t)(\gamma) = [\kappa](\gamma) \prod_{j=1}^r \text{wt}_{\text{Fock}}(t)(\mathbf{w}_j).$$

Since $|\kappa(\gamma)| = 1$, taking logarithms of absolute values and using (6.8) gives

$$\log |[\text{wt}](t)(\gamma)| = \left(\sum_{j=1}^r \text{wt}_{\text{Fock}}^\dagger(\mathbf{w}_j) \right) \log t + O(1).$$

Since the weights are positive for $t \gg 1$, the absolute values may be dropped, and therefore

$$[\text{wt}^\dagger] = [\text{wt}_{\text{Fock}}^\dagger],$$

which proves the theorem. $\square$

Example 6.2. We verify Theorem 6.1 for our running example. Recall the positive Puiseux weights

$$X_1(t) = t, \quad X_2(t) = t, \quad X_3(t) = 1, \quad \rho_z(t) = t, \quad \rho_w(t) = 1$$

from Example 3.1 with tropical weights

$$X_1^\dagger = 1, \quad X_2^\dagger = 1, \quad X_3^\dagger = 0, \quad \rho_z^\dagger = 1, \quad \rho_w^\dagger = 0.$$

The tropical spectral curve is shown in Figure 7. From Example 2.9, the standard divisor is

$$D(t) = \left(-\frac{X_1 X_2 X_3}{\rho_z}, \frac{1 + X_1 X_2 X_3}{X_1 \rho_w} \right) = (-t, t + t^{-1}),$$

which tropicalizes to the point $D^\dagger = (1, 1)$. Next, we compute the tropical angle map. Again from Example 2.9, we have

$$C_{\alpha_1} = \rho_z = t, \quad C_{\alpha_2} = \frac{\rho_z}{X_1 X_2} = t^{-1}.$$

Since $[\alpha_1] = [\alpha_2] = (-1, 0)$, the lines containing the unbounded edges are

$$x + C_{\alpha_i}^\dagger = 0.$$

Since $C_{\alpha_1}^\dagger = 1, C_{\alpha_2}^\dagger = -1$, we get

$$\alpha_1^\dagger = (-1, 0), \quad \alpha_2^\dagger = (1, 0).$$

Thus, the tropical angle map is

$$\alpha_1^\dagger = (-1, 0), \quad \alpha_2^\dagger = (1, 0), \quad \beta^\dagger = (-2, 1), \quad \gamma^\dagger = (2, 1).$$

We choose as spanning tree T the complement of the edge $\beta^\dagger \gamma^\dagger$, and take the basepoint to be $\zeta_0^\dagger = \beta^\dagger$. The degeneration-adapted tropical B -cycle is the bounded face, and so the period matrix is

$$\Pi^{\text{deg}, \dagger} = 4 + 1 + 2 + 1 = 8,$$

and the tropical theta function is

$$\theta^\dagger(x \mid 8) = \max_{m \in \mathbb{Z}} \{-4m^2 - mx\}.$$

The values needed below are

$$\theta^\dagger(-3 \mid 8) = 0, \quad \theta^\dagger(-2 \mid 8) = 0, \quad \theta^\dagger(2 \mid 8) = 0, \quad \theta^\dagger(5 \mid 8) = 1;$$

see Figure 10.

We have $\mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(D^\dagger) = -5$, and by Proposition 5.6,

$$\Delta^{\text{deg}, \dagger} = -\frac{1}{2} \Pi^{\text{deg}, \dagger} = -4.$$

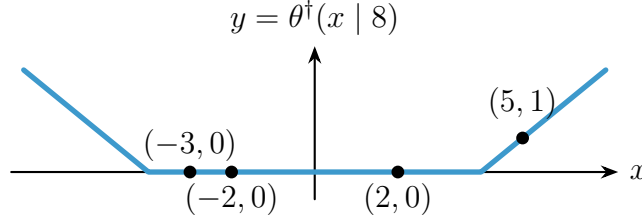


FIGURE 10. The tropical theta function $\theta^\dagger(x \mid 8) = \max_{m \in \mathbb{Z}} \{-4m^2 - mx\}$.

The boundary of $\mathbf{f}_1$ decomposes into three wedges $\partial \mathbf{f}_1 = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$, whose associated zig-zag paths are

	α	β_-	β_+
$\mathbf{w}_1$	γ	β	α_1
$\mathbf{w}_2$	β	α_1	γ
$\mathbf{w}_3$	α_2	γ	β

We next compute the tropical discrete Abel map for the faces appearing around $\mathbf{f}_1$. Recall that we normalize by $\mathbf{d}^\dagger(\mathbf{w}_1) = 0$. We have $\mathbf{d}^\dagger(\mathbf{f}_1) = \beta^\dagger$, and hence $\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_1)) = 0$.

For the wedges, the relevant values of the tropical discrete Abel map are as follows:

wedge	$\mathbf{f}_-$	$\mathbf{d}^\dagger(\mathbf{f}_-)$	$\mathbf{f}_+$	$\mathbf{d}^\dagger(\mathbf{f}_+)$
$\mathbf{w}_1$	$\mathbf{f}_2 + (-1, 0)$	$\gamma^\dagger$	$\mathbf{f}_3 + (0, -1)$	$\beta^\dagger + \gamma^\dagger - \alpha_1^\dagger$
$\mathbf{w}_2$	$\mathbf{f}_4 + (-1, 0)$	$2\beta^\dagger - \alpha_1^\dagger$	$\mathbf{f}_2$	$2\beta^\dagger - \gamma^\dagger$
$\mathbf{w}_3$	$\mathbf{f}_4$	$\beta^\dagger + \alpha_2^\dagger - \gamma^\dagger$	$\mathbf{f}_3$	$\alpha_2^\dagger$

The pairs of integers denotes which $\mathbb{Z}^2$ -translate of the face in the fundamental domain is being considered. Applying the tropical Abel map gives

$$\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\alpha_1^\dagger) = -1, \quad \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\alpha_2^\dagger) = -3, \quad \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\beta^\dagger) = 0, \quad \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\gamma^\dagger) = -4,$$

and therefore,

wedge	$\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_-))$	$\mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_+))$
$\mathbf{w}_1$	-4	-3
$\mathbf{w}_2$	1	4
$\mathbf{w}_3$	1	-3

We now compute the three wedge weights. For $\mathbf{w}_1$, the adjacent faces are $\mathbf{f}_- = \mathbf{f}_2, \mathbf{f}_+ = \mathbf{f}_3$, and the associated zig-zag paths are $(\alpha, \beta_-, \beta_+) = (\gamma, \beta, \alpha_1)$. Hence, the arguments of the theta function are

$$\begin{aligned} \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_2)) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger) + \Delta^{\deg, \dagger} &= -4 - (-5) - 4 = -3, \\ \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_3)) - \mu_{\zeta_0^\dagger}^{\deg, \dagger}(\mathbf{d}^\dagger) + \Delta^{\deg, \dagger} &= -3 - (-5) - 4 = -2. \end{aligned}$$

The differential $\omega_{\alpha_1^\dagger - \beta^\dagger}^\dagger$ is the unit flow along the path $(\beta^\dagger \rightarrow \alpha_1^\dagger)$ and the integral $\int_{\zeta_0^\dagger}^{\gamma^\dagger} \omega_{\alpha_1^\dagger - \beta^\dagger}^\dagger$ is along the path $(\beta^\dagger \rightarrow \alpha_1^\dagger \rightarrow \alpha_2^\dagger \rightarrow \gamma^\dagger)$, so

$$\int_{\zeta_0^\dagger}^{\gamma^\dagger} \omega_{\alpha_1^\dagger - \beta^\dagger}^\dagger = 1.$$

Therefore,

$$\begin{aligned} \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}_1) &= \theta^\dagger(-3 \mid 8) - \theta^\dagger(-2 \mid 8) - \int_{\zeta_0^\dagger}^{\gamma^\dagger} \omega_{\alpha_1^\dagger - \beta^\dagger}^\dagger \\ &= 0 - 0 - 1 \\ &= -1. \end{aligned}$$

For $\mathfrak{w}_2$, the adjacent faces are $\mathbf{f}_- = \mathbf{f}_4, \mathbf{f}_+ = \mathbf{f}_2$, and the associated zig-zag paths are $(\alpha, \beta_-, \beta_+) = (\beta, \alpha_1, \gamma)$. Hence, the arguments of the theta function are

$$\begin{aligned} \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_4)) - \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(D^\dagger) + \Delta^{\text{deg}, \dagger} &= 1 - (-5) - 4 = 2, \\ \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_2)) - \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(D^\dagger) + \Delta^{\text{deg}, \dagger} &= 4 - (-5) - 4 = 5. \end{aligned}$$

Since $\alpha^\dagger = \beta^\dagger = \zeta_0^\dagger$, the integration path is constant, and therefore

$$\int_{\zeta_0^\dagger}^{\beta^\dagger} \omega_{\gamma^\dagger - \alpha_1^\dagger}^\dagger = 0.$$

Thus,

$$\begin{aligned} \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}_2) &= \theta^\dagger(2 \mid 8) - \theta^\dagger(5 \mid 8) - \int_{\zeta_0^\dagger}^{\beta^\dagger} \omega_{\gamma^\dagger - \alpha_1^\dagger}^\dagger \\ &= 0 - 1 - 0 \\ &= -1. \end{aligned}$$

For $\mathfrak{w}_3$, the adjacent faces are $\mathbf{f}_- = \mathbf{f}_4, \mathbf{f}_+ = \mathbf{f}_3$, and the associated zig-zag paths are $(\alpha, \beta_-, \beta_+) = (\alpha_2, \gamma, \beta)$. Hence, the arguments of the theta function are

$$\begin{aligned} \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_4)) - \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(D^\dagger) + \Delta^{\text{deg}, \dagger} &= 1 - (-5) - 4 = 2, \\ \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(\mathbf{d}^\dagger(\mathbf{f}_3)) - \mu_{\zeta_0^\dagger}^{\text{deg}, \dagger}(D^\dagger) + \Delta^{\text{deg}, \dagger} &= -3 - (-5) - 4 = -2. \end{aligned}$$

The differential $\omega_{\beta^\dagger - \gamma^\dagger}^\dagger$ is the unit flow along the path $(\gamma^\dagger \rightarrow \alpha_2^\dagger \rightarrow \alpha_1^\dagger \rightarrow \beta^\dagger)$, while the integration path from $\zeta_0^\dagger = \beta^\dagger$ to $\alpha_2^\dagger$ is the path $(\beta^\dagger \rightarrow \alpha_1^\dagger \rightarrow \alpha_2^\dagger)$. Hence,

$$\int_{\zeta_0^\dagger}^{\alpha_2^\dagger} \omega_{\beta^\dagger - \gamma^\dagger}^\dagger = -3.$$

Therefore,

$$\begin{aligned} \text{wt}_{\text{Fock}}^\dagger(\mathfrak{w}_3) &= \theta^\dagger(2 \mid 8) - \theta^\dagger(-2 \mid 8) - \int_{\zeta_0^\dagger}^{\alpha_2^\dagger} \omega_{\beta^\dagger - \gamma^\dagger}^\dagger \\ &= 0 - 0 - (-3) \\ &= 3. \end{aligned}$$

The sum of the wedge weights is

$$[\text{wt}_{\text{Fock}}^\dagger](\partial \mathbf{f}_1) = -1 - 1 + 3 = 1 = X_1^\dagger.$$

Thus, the tropical Fock formula recovers the original tropical face weight around $\mathbf{f}_1$.

7. A TROPICAL FAY'S TRISECANT IDENTITY

In this section, we apply our asymptotic results to prove a tropical version of Fay's trisecant identity. Tropical versions of Fay's identity were first obtained by Inoue–Takenawa [IT09a] for hyperelliptic tropical curves and subsequently generalized by Inoue–Iwao [II11] to general smooth planar tropical curves. We first rewrite the classical identity in a form involving normalized differentials of the third kind.

Proposition 7.1 (Fay's trisecant identity [Fay73, p. 34, Eq. (45)]). *Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ be four distinct points of a compact Riemann surface Σ , and choose lifts of these points to the universal cover. Let $\xi \in \mathbb{C}^g$ and let*

$$\begin{aligned} F_1 &:= \theta(\xi + \eta_{13} \mid \Pi) \theta(\xi + \eta_{24} \mid \Pi) \exp \left(\int_{\zeta_1}^{\zeta_3} \omega_{\zeta_2 - \zeta_4} \right) \\ F_2 &:= \theta(\xi + \eta_{23} \mid \Pi) \theta(\xi + \eta_{14} \mid \Pi) \exp \left(\int_{\zeta_2}^{\zeta_3} \omega_{\zeta_1 - \zeta_4} \right) \\ F_3 &:= \theta(\xi \mid \Pi) \theta(\xi + \eta_{13} + \eta_{24} \mid \Pi), \end{aligned}$$

where $\eta_{ij} := \mu_{\zeta_0}(\zeta_j - \zeta_i)$. Then,

$$F_3 = F_1 + F_2.$$

We now tropicalize Proposition 7.1.

Theorem 7.2 (Tropical Fay's trisecant identity). *Let $(\Sigma^\dagger, \ell, \mathbf{g})$ be an abstract tropical curve arising from a degeneration as in Section 3.3, and let $\zeta_1^\dagger, \zeta_2^\dagger, \zeta_3^\dagger, \zeta_4^\dagger \in V(\Sigma^\dagger) \cap \partial f_0$ be four vertices on the boundary of the outer face, labeled so that they occur weakly in the cyclic order*

$$\zeta_1^\dagger \leq \zeta_3^\dagger \leq \zeta_2^\dagger \leq \zeta_4^\dagger \leq \zeta_1^\dagger$$

along ∂f_0 . By weakly, we mean that the points are allowed to coincide. Choose an edge e_0 on the arc of ∂f_0 between $\zeta_4^\dagger$ and $\zeta_1^\dagger$ containing none of the other points, and choose the spanning tree T so that it contains every edge of ∂f_0 except e_0 . Let $\xi^\dagger \in \mathbb{R}^{g^\dagger}$ and let

$$\begin{aligned} F_1^\dagger &:= \theta^\dagger(\xi^\dagger + \eta_{13}^\dagger \mid \Pi^{\text{deg}, \dagger}) + \theta^\dagger(\xi^\dagger + \eta_{24}^\dagger \mid \Pi^{\text{deg}, \dagger}) - \int_{\zeta_1^\dagger}^{\zeta_3^\dagger} \omega_{\zeta_2^\dagger - \zeta_4^\dagger}^{\text{deg}, \dagger}, \\ F_2^\dagger &:= \theta^\dagger(\xi^\dagger + \eta_{23}^\dagger \mid \Pi^{\text{deg}, \dagger}) + \theta^\dagger(\xi^\dagger + \eta_{14}^\dagger \mid \Pi^{\text{deg}, \dagger}) - \int_{\zeta_2^\dagger}^{\zeta_3^\dagger} \omega_{\zeta_1^\dagger - \zeta_4^\dagger}^{\text{deg}, \dagger}, \\ F_3^\dagger &:= \theta^\dagger(\xi^\dagger \mid \Pi^{\text{deg}, \dagger}) + \theta^\dagger(\xi^\dagger + \eta_{13}^\dagger + \eta_{24}^\dagger \mid \Pi^{\text{deg}, \dagger}), \end{aligned}$$

where $\eta_{ij}^\dagger := \mu_{\zeta_0^\dagger}^\dagger(\zeta_j^\dagger - \zeta_i^\dagger)$, and all tropical integrals are taken along the corresponding unique paths in the cut tropical curve $\Sigma_{\text{cut}}^\dagger$. Then,

$$F_3^\dagger = \max\{F_1^\dagger, F_2^\dagger\}.$$

Proof. Let $\Sigma(t) \rightarrow \Sigma(\infty)$ be the degeneration as in Section 4.1. For $1 \leq i \leq 4$, choose distinct real points

$$\zeta_i \in \Sigma_{v_i}^{\text{core}}(\infty)$$

such that the points $\zeta_i(t) = \Phi_{v_i}(\zeta_i; t^{-1})$ lie on the outer oval $B_0^{\text{ov}}(t)$ in the same cyclic order, and set $\zeta_i^\dagger := v_i$. Set

$$\boldsymbol{\eta}_{ij}(t) := \int_{\zeta_i(t)}^{\zeta_j(t)} \omega^{\text{deg}}(t).$$

After cutting at e_0 , the unique path in T between any two of the $\zeta_i^\dagger$ is the corresponding arc of $\partial f_0 \setminus e_0$. We choose its degeneration-adapted lift to be the corresponding arc of $B_0^{\text{ov}}(t)$, and take the classical integral along this arc. By the cyclic ordering of the marked points, the arc from $\zeta_1(t)$ to $\zeta_3(t)$ is disjoint from $\zeta_2(t)$ and $\zeta_4(t)$, while the arc from $\zeta_2(t)$ to $\zeta_3(t)$ is disjoint from $\zeta_1(t)$ and $\zeta_4(t)$.

By Part (1) of Proposition 2.6, together with the change-of-basis from the oval-adapted to the degeneration-adapted basis, we have

$$\boldsymbol{\eta}_{ij}(t) \in \mathbb{R}^g.$$

Let $\boldsymbol{\xi}(t) \in \mathbb{R}^g$ be a family satisfying

$$\boldsymbol{\xi}(t) = - \begin{pmatrix} 0 \\ \boldsymbol{\xi}^\dagger \end{pmatrix} \log t + O(1)$$

for some $\boldsymbol{\xi}^\dagger \in \mathbb{R}^{g^\dagger}$. Then all the arguments of the theta functions appearing in Proposition 7.1 are real for $t \gg 1$. By Proposition 5.4,

$$\boldsymbol{\eta}_{ij}(t) = - \begin{pmatrix} 0 \\ \boldsymbol{\eta}_{ij}^\dagger \end{pmatrix} \log t + O(1).$$

Therefore, by Proposition 5.11,

$$\log \theta(\boldsymbol{\xi}(t) + \boldsymbol{\eta}_{ij}(t) \mid \Pi^{\text{deg}}(t)) = \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{ij}^\dagger \mid \Pi^{\text{deg}, \dagger}) \log t + O(1),$$

$$\log \theta(\boldsymbol{\xi}(t) \mid \Pi^{\text{deg}}(t)) = \theta^\dagger(\boldsymbol{\xi}^\dagger \mid \Pi^{\text{deg}, \dagger}) \log t + O(1),$$

$$\log \theta(\boldsymbol{\xi}(t) + \boldsymbol{\eta}_{13}(t) + \boldsymbol{\eta}_{24}(t) \mid \Pi^{\text{deg}}(t)) = \theta^\dagger(\boldsymbol{\xi}^\dagger + \boldsymbol{\eta}_{13}^\dagger + \boldsymbol{\eta}_{24}^\dagger \mid \Pi^{\text{deg}, \dagger}) \log t + O(1).$$

By Proposition 5.5,

$$\begin{aligned} \int_{\zeta_1(t)}^{\zeta_3(t)} \omega_{\zeta_2(t)-\zeta_4(t)}^{\text{deg}}(t) &= - \left(\int_{\zeta_1^\dagger}^{\zeta_3^\dagger} \omega_{\zeta_2^\dagger-\zeta_4^\dagger}^{\text{deg}, \dagger} \right) \log t + O(1), \\ \int_{\zeta_2(t)}^{\zeta_3(t)} \omega_{\zeta_1(t)-\zeta_4(t)}^{\text{deg}}(t) &= - \left(\int_{\zeta_2^\dagger}^{\zeta_3^\dagger} \omega_{\zeta_1^\dagger-\zeta_4^\dagger}^{\text{deg}, \dagger} \right) \log t + O(1). \end{aligned}$$

Let $F_i(t)$, $1 \leq i \leq 3$, denote the three terms in Proposition 7.1. Since the arguments of the theta functions are real and the period matrix is real by Part (2) of Proposition 2.6 together with change-of-basis, every term in the series expansions of the theta functions are positive, and therefore the theta functions are also positive. Moreover, the exponential factors are also positive since the third kind differentials are real and we are integrating along real paths. Therefore, $F_1(t), F_2(t), F_3(t) > 0$. Combining the asymptotics above gives

$$\log F_i(t) = F_i^\dagger \log t + O(1), \quad 1 \leq i \leq 3.$$

Since each $F_i(t)$ is positive, there is no cancellation of leading terms and so

$$F_3^\dagger = \max\{F_1^\dagger, F_2^\dagger\}.$$

□

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